Introduction to Quantum Information

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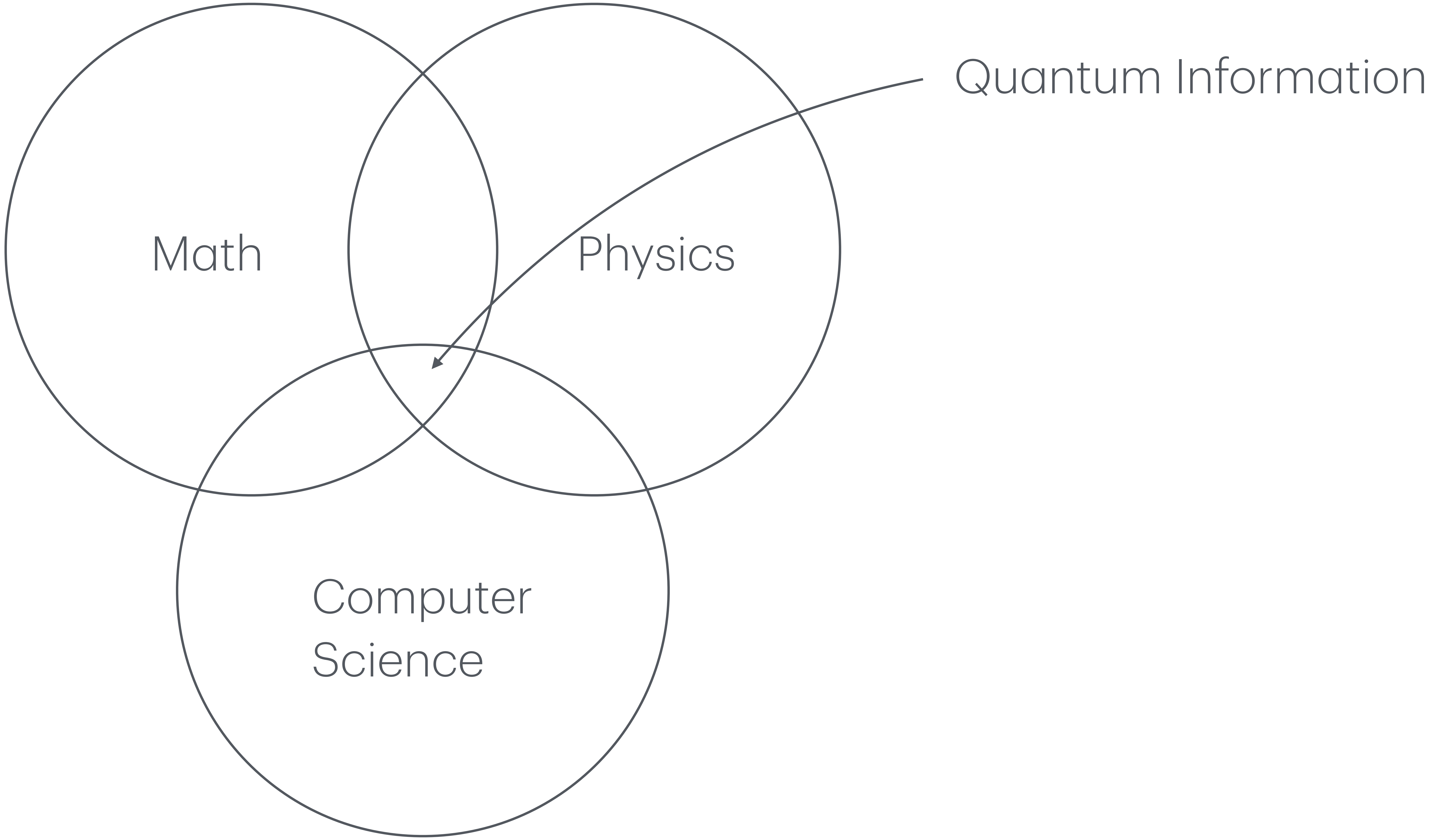
Quantum Information



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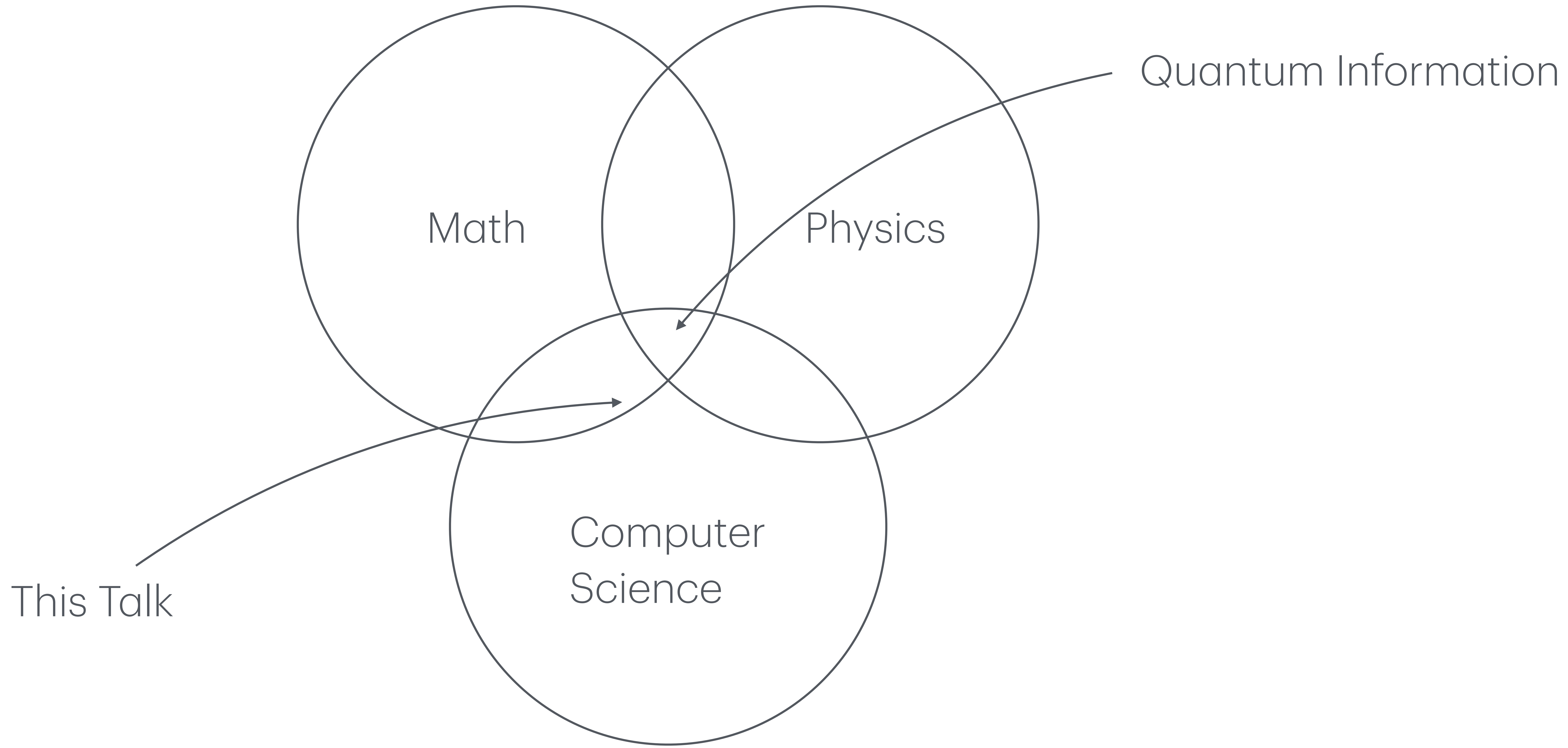

Linear Algebra Review

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- Matrix conjugate: $M^\dagger_{i,j} = M^*_{j,i}$ where $M \in \mathbb{C}^{n \times n}$

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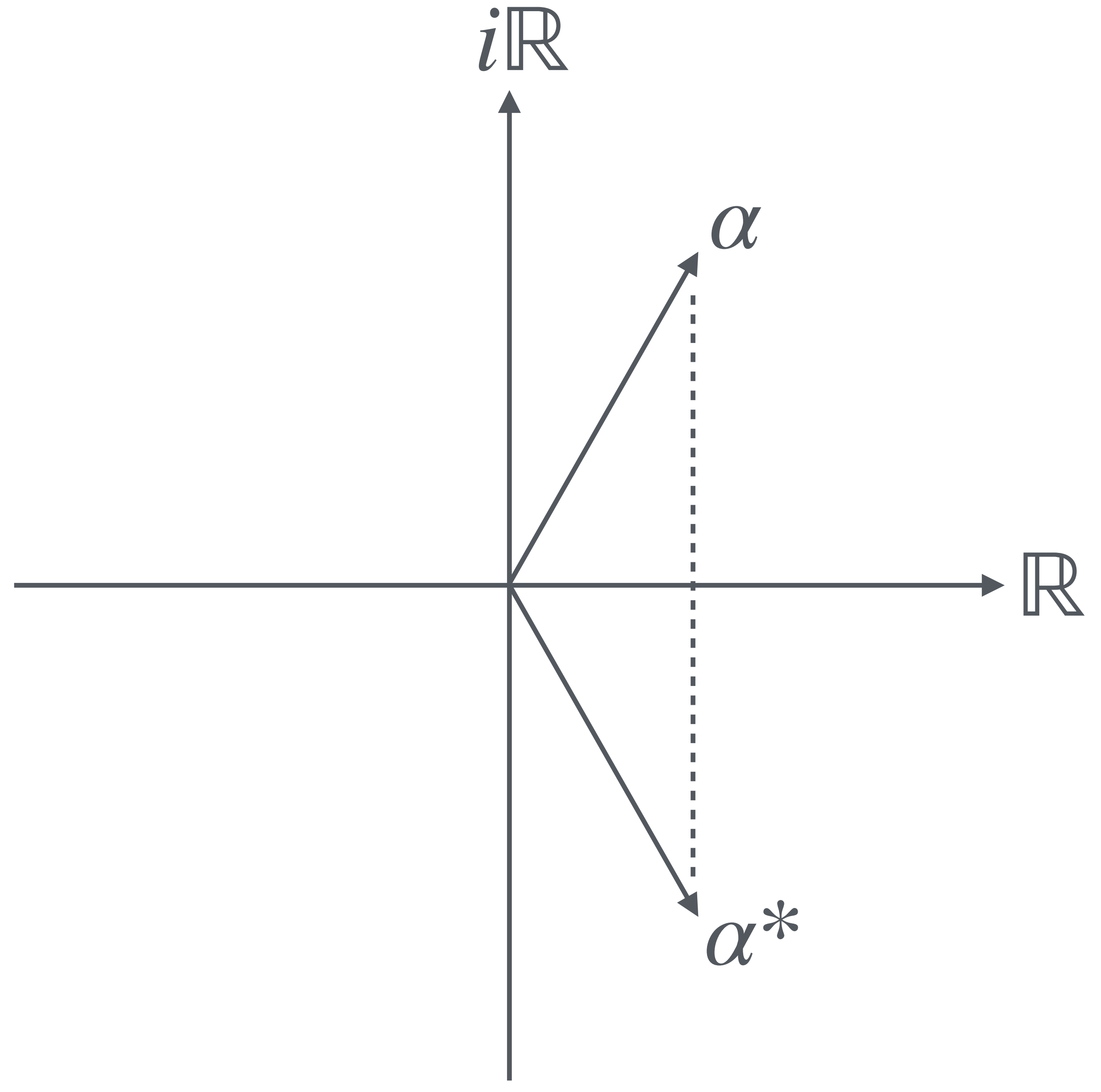

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- Of course nothing stops us from introducing more qubits to the system…

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- Any (pure) quantum state can be written as:

$$|\psi> = \sum_x \alpha_x |x> \qquad \sum_x |\alpha_x|^2 = 1$$

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    - States that cannot be decomposed into tensors are called entangled

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Manipulating Quantum States

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- For convenience, we have set the Planck constant to 1

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- A unitary matrix preserves norms, and thus it maps quantum states to quantum states

$$\|U\psi\|^2 = < U\psi | U\psi > = < \psi | U^\dagger U | \psi > = < \psi | \psi > = 1$$

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  - This is WLOG, since a basis change is a unitary operation

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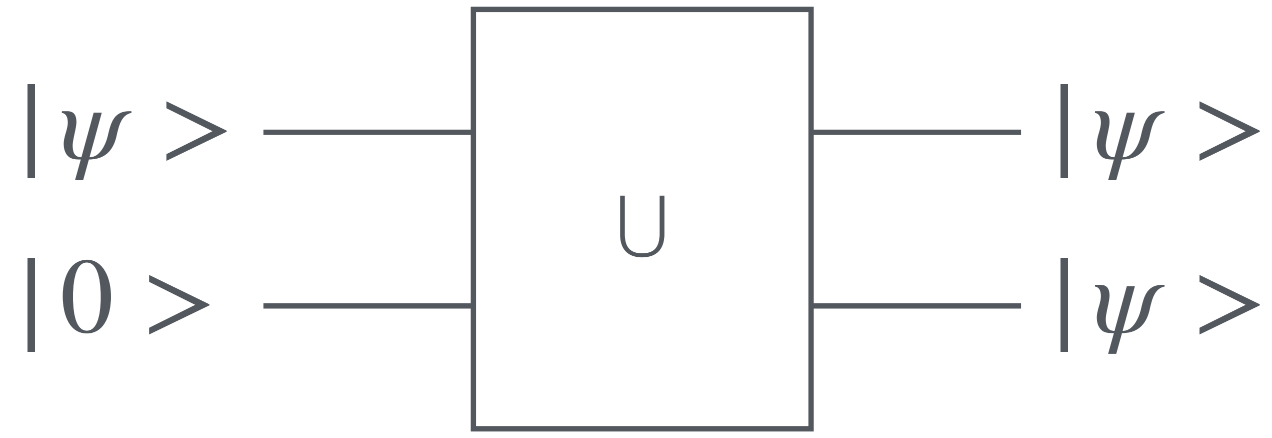

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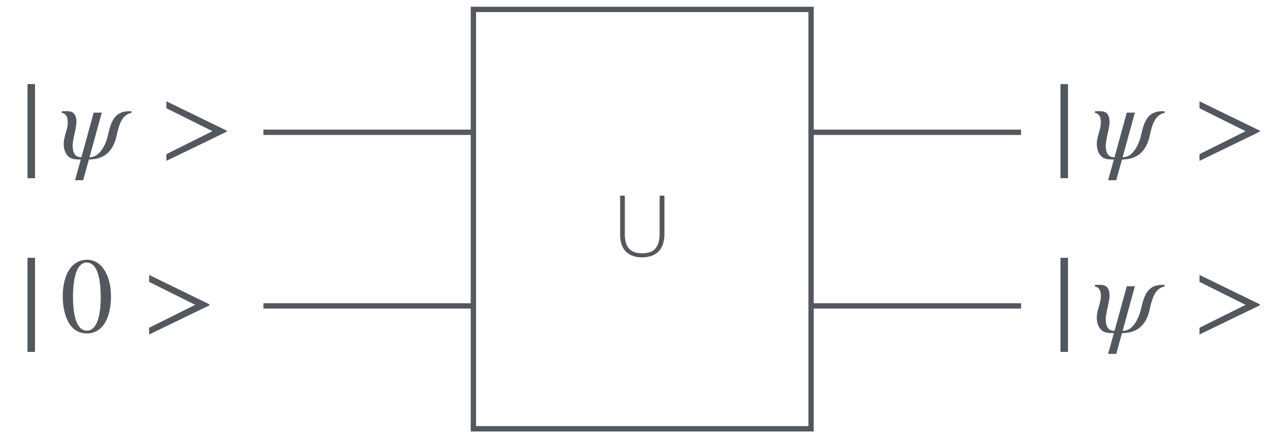

- This implements the mapping

$$U(\alpha|0> + \beta|1>) \otimes |0> \; = (\alpha|0> + \beta|1>) \otimes (\alpha|0> + \beta|1>)$$
$$= \alpha^2|00> + \alpha\beta|01> + \alpha\beta|10> + \beta^2|11>$$

which is not linear, and in particular not unitary

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- Now that we know the rules, let us see how/what we can compute in this model

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- So the question is, what kind of functions admit a “reversible” implementation?

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- THEOREM: If f can be implemented using s-many NAND gates, it can be implemented using O(s)-many Toffoli gates

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- In complexity theory terms: P ⊆ BQP

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  - Proof Sketch: Keep track of the evolution of the amplitudes. To make the space polynomial, use Feynman path integrals!
- The catch of course is that this simulation is *inefficient*
- The question to ask is: Are there quantum computations that we cannot simulate *efficiently* with classical computers?

Quantum Circuits

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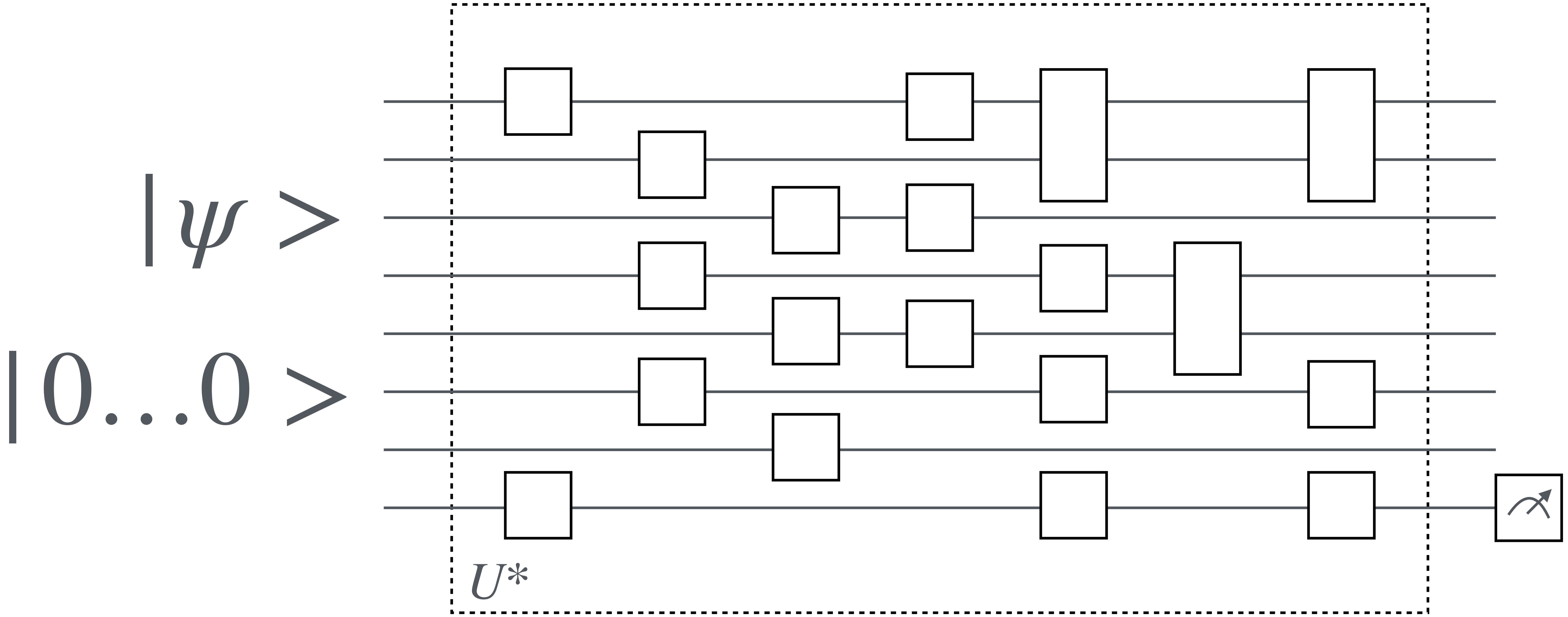

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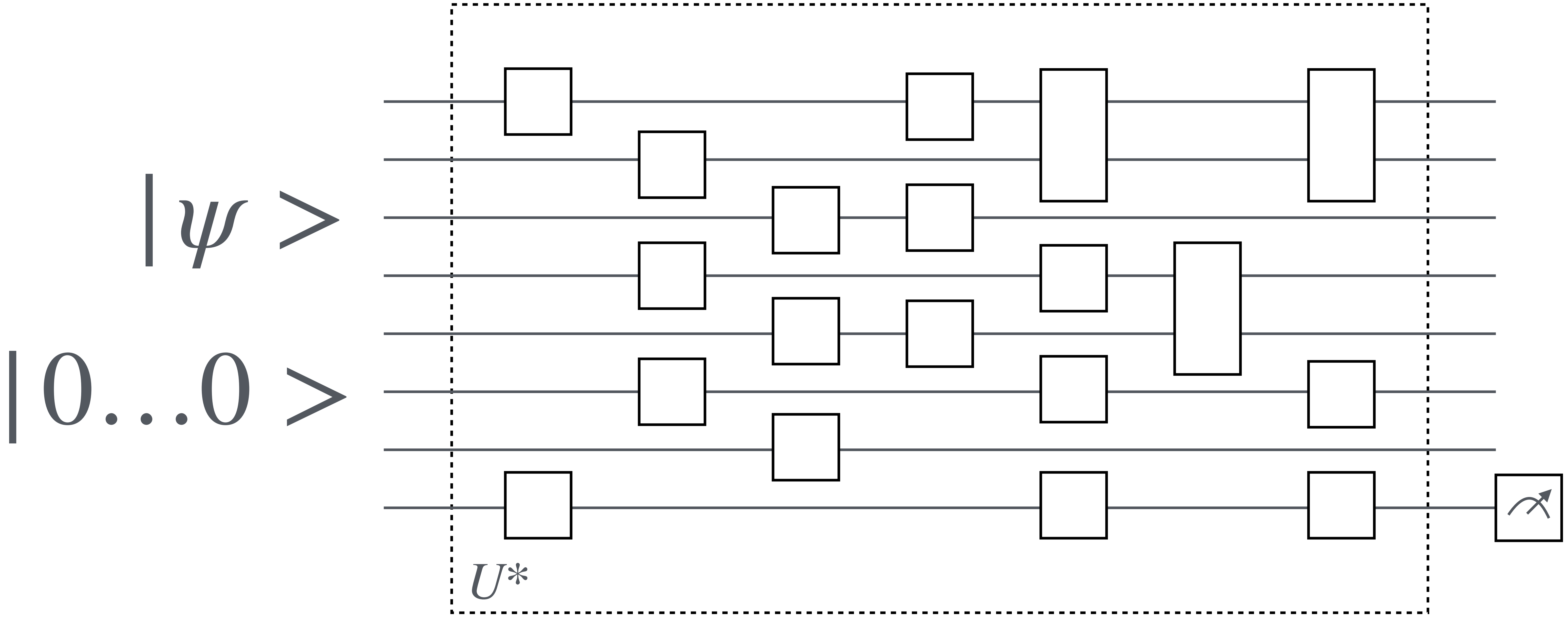

- Efficiency is measured in terms of number of gates (assuming each gate is a constant-size unitary)

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$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H | b > = \frac{1}{\sqrt{2}} (| 0 > + (-1)^b | 1 >)$$

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$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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- The Solovay-Kitaev theorem bounds the number of gates (for *any* universal gate set) needed to approximate any (constant-dimension) unitary up to ε precision, by poly-log$(1/\varepsilon)$

Quantum Fourier Transform

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- The more general q-ary QFT is also efficiently computable, where $\omega_q = e^{2\pi i/q}$ is the q-th root of unity

$$QFT_q|x> = \frac{1}{\sqrt{N}}\sum_y \omega_q^{x\cdot y}|y>$$

Bonus: The CHSH Game

The EPR Paradox

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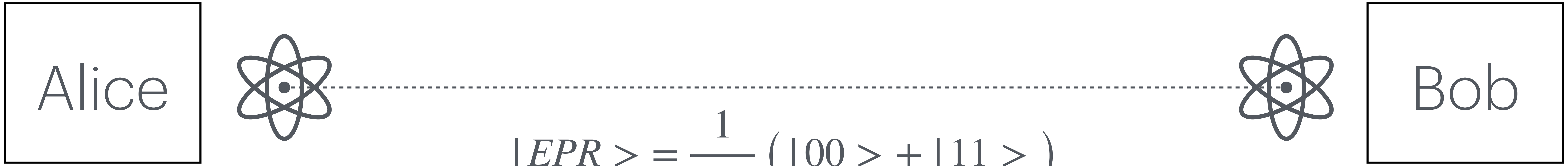

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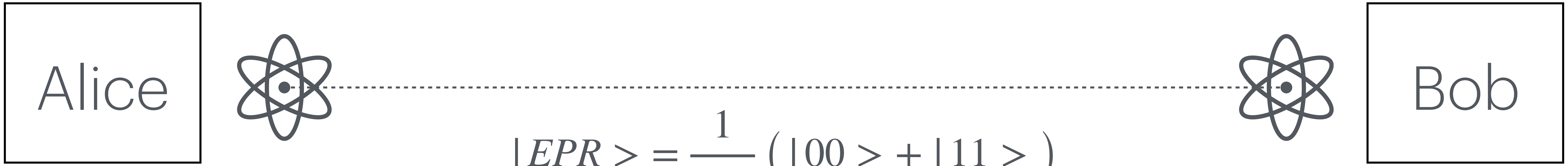

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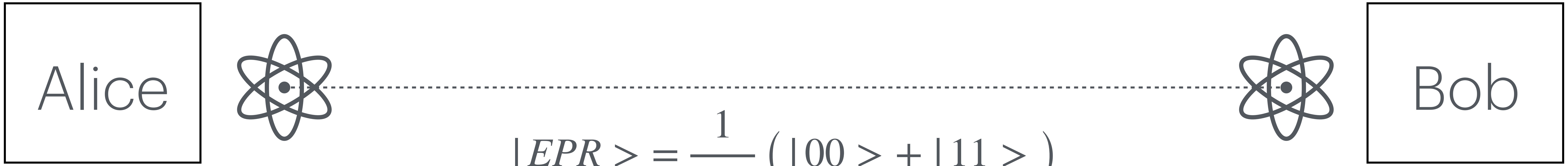

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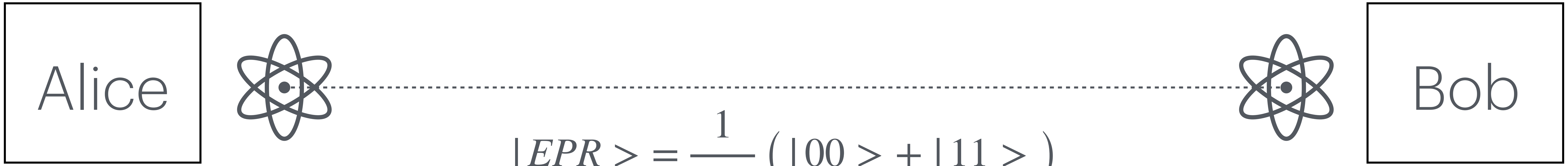

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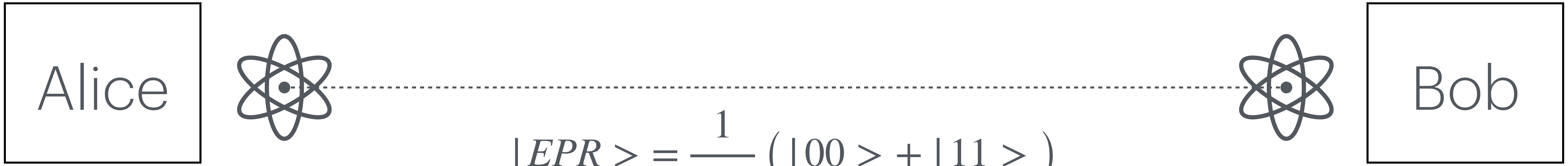

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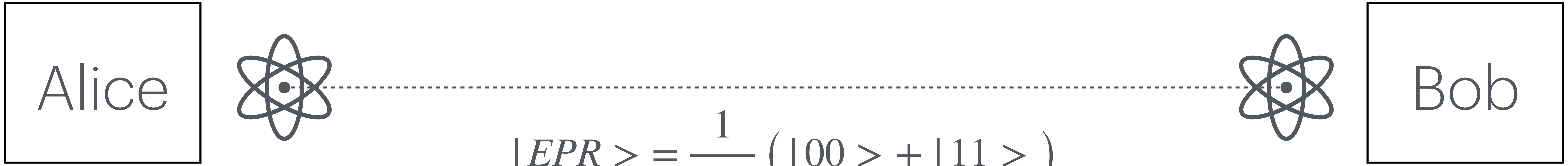

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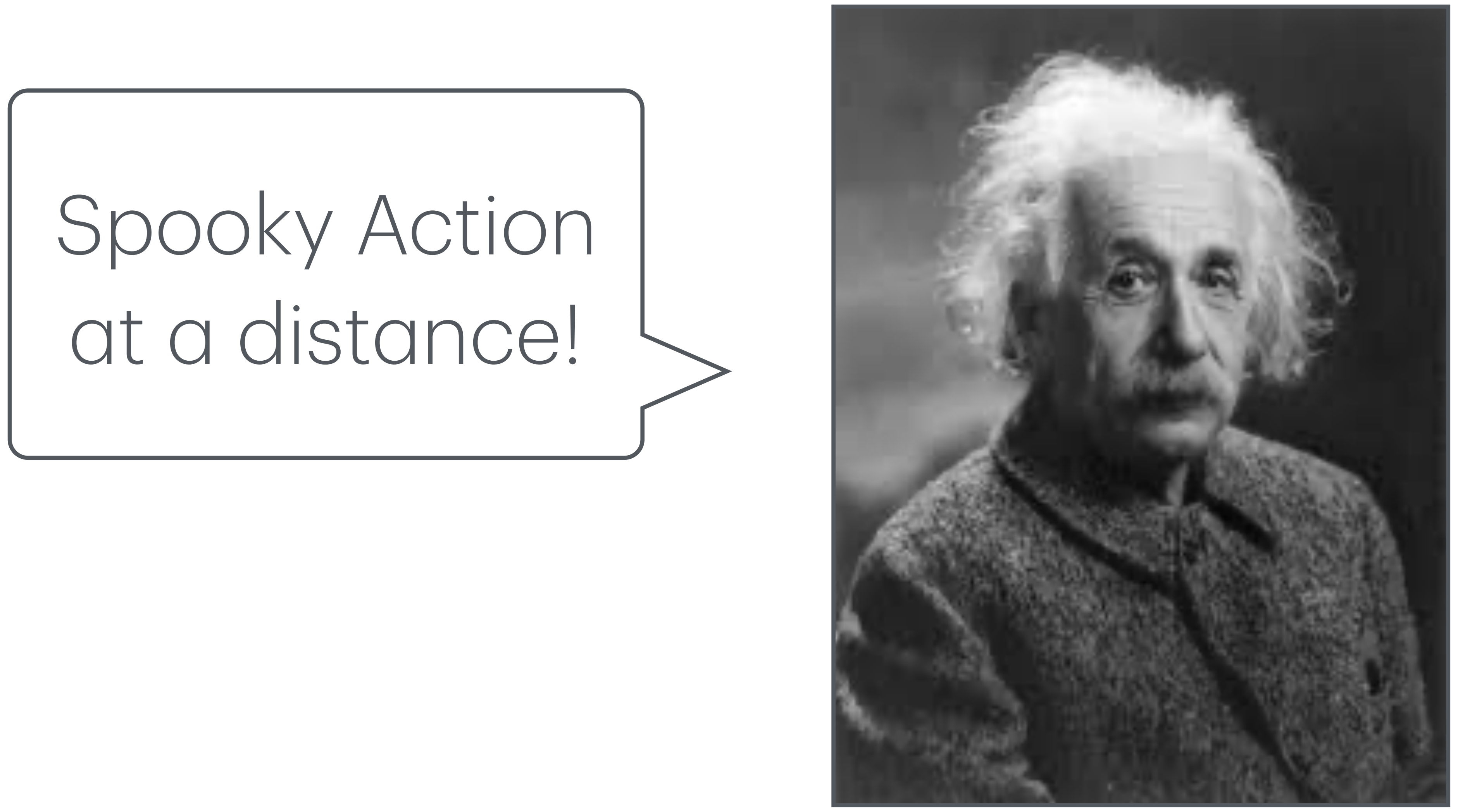

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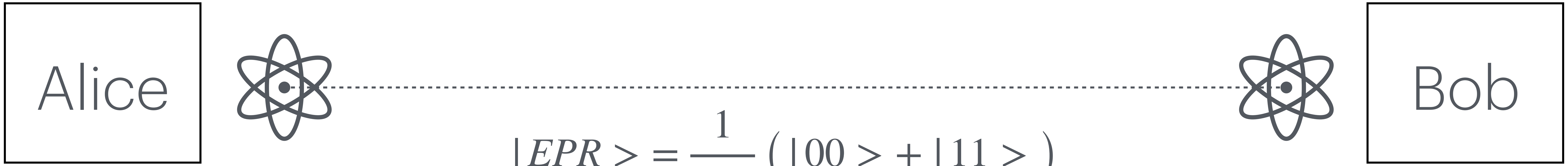

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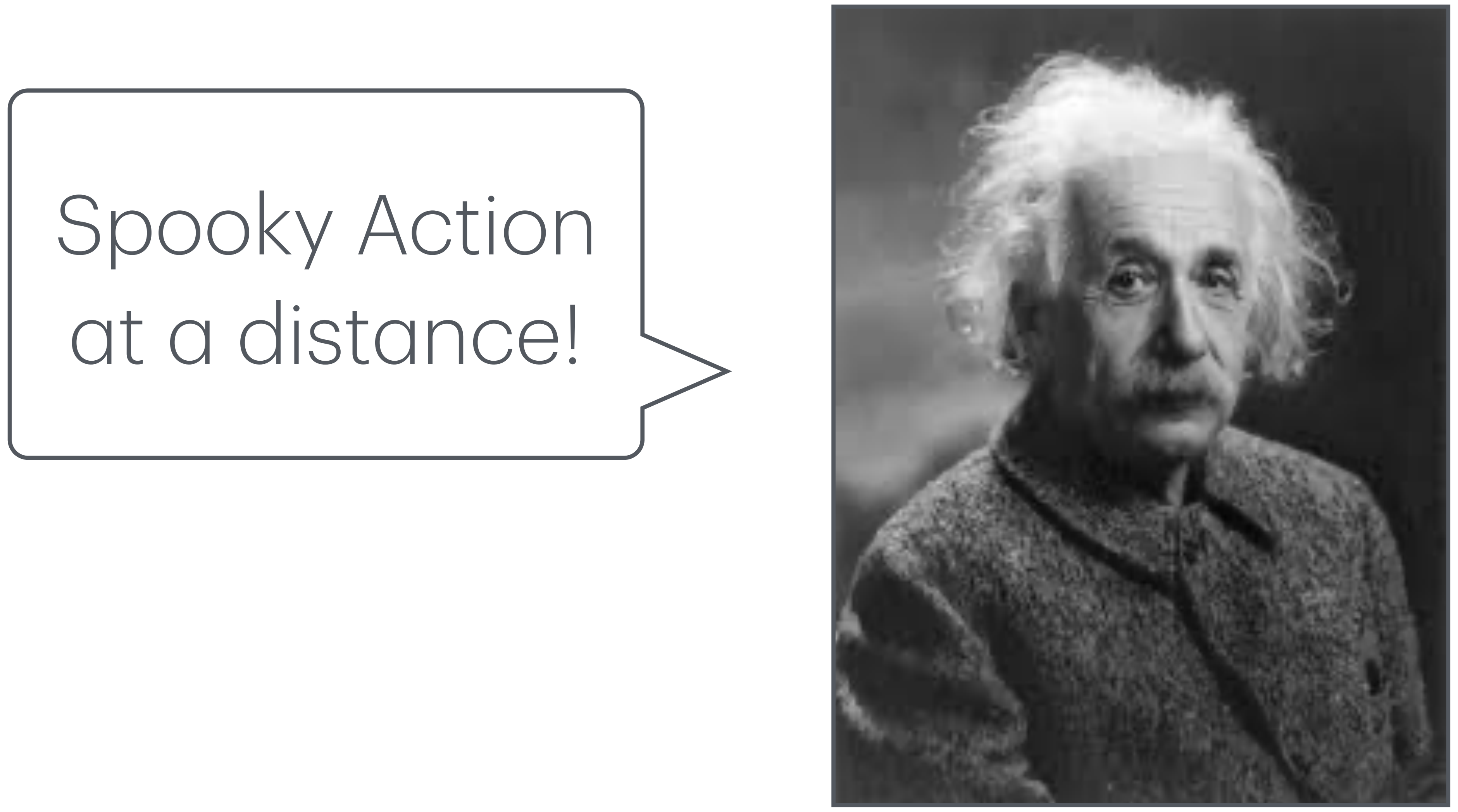

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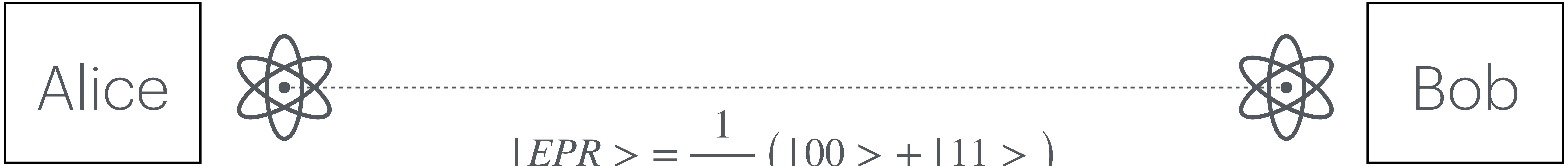

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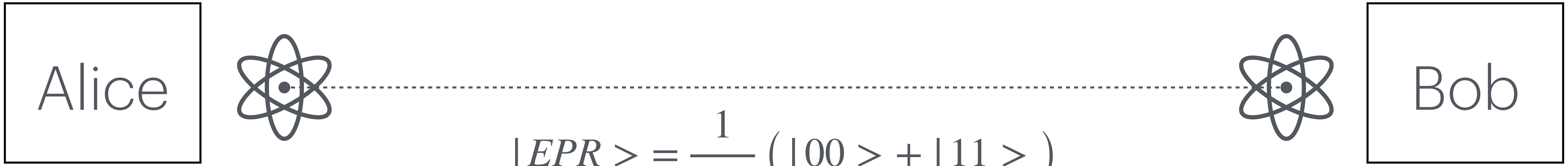

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Quantum Mechanics

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- PROBLEM: The probabilities are identical in both cases!

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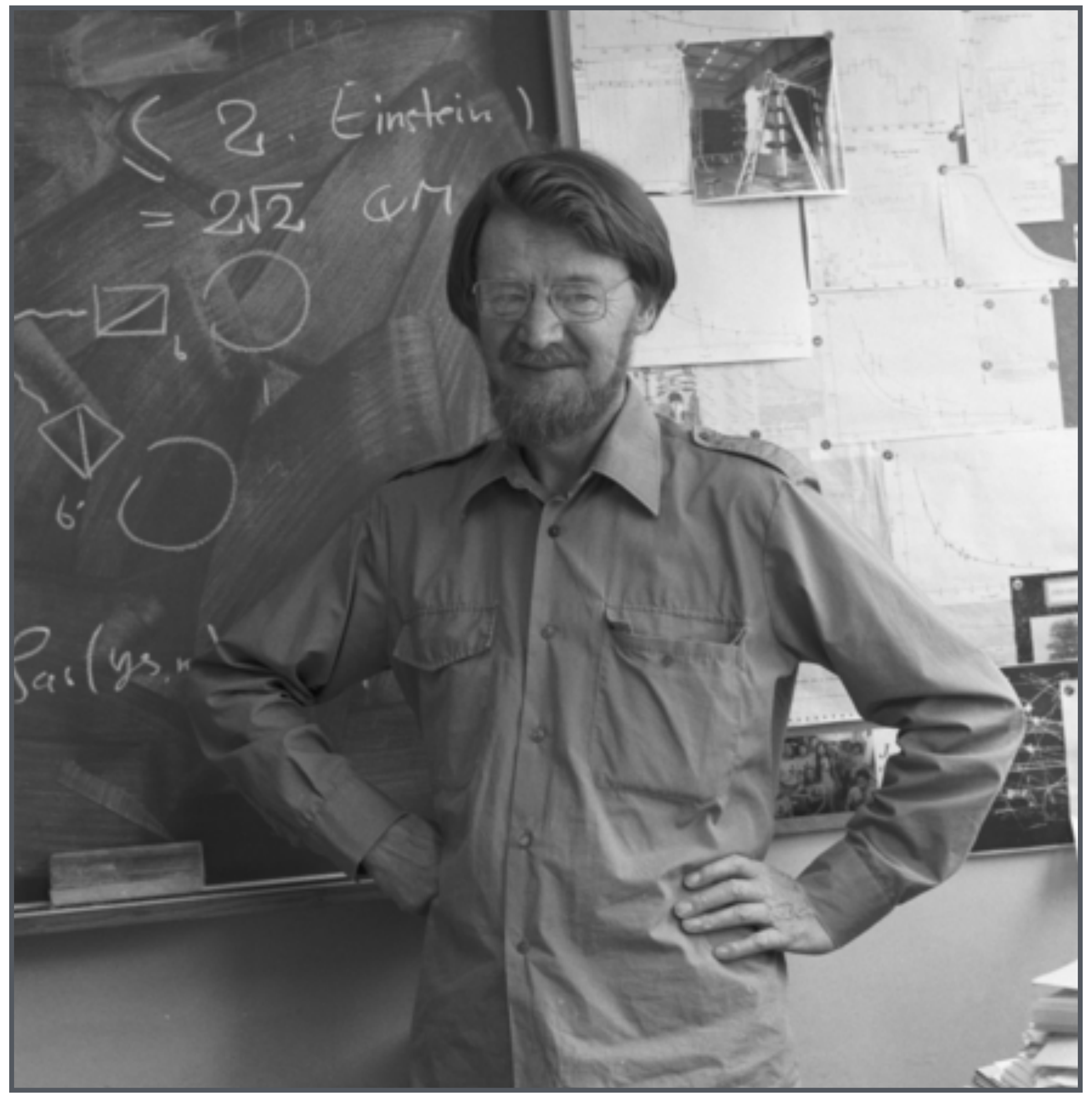

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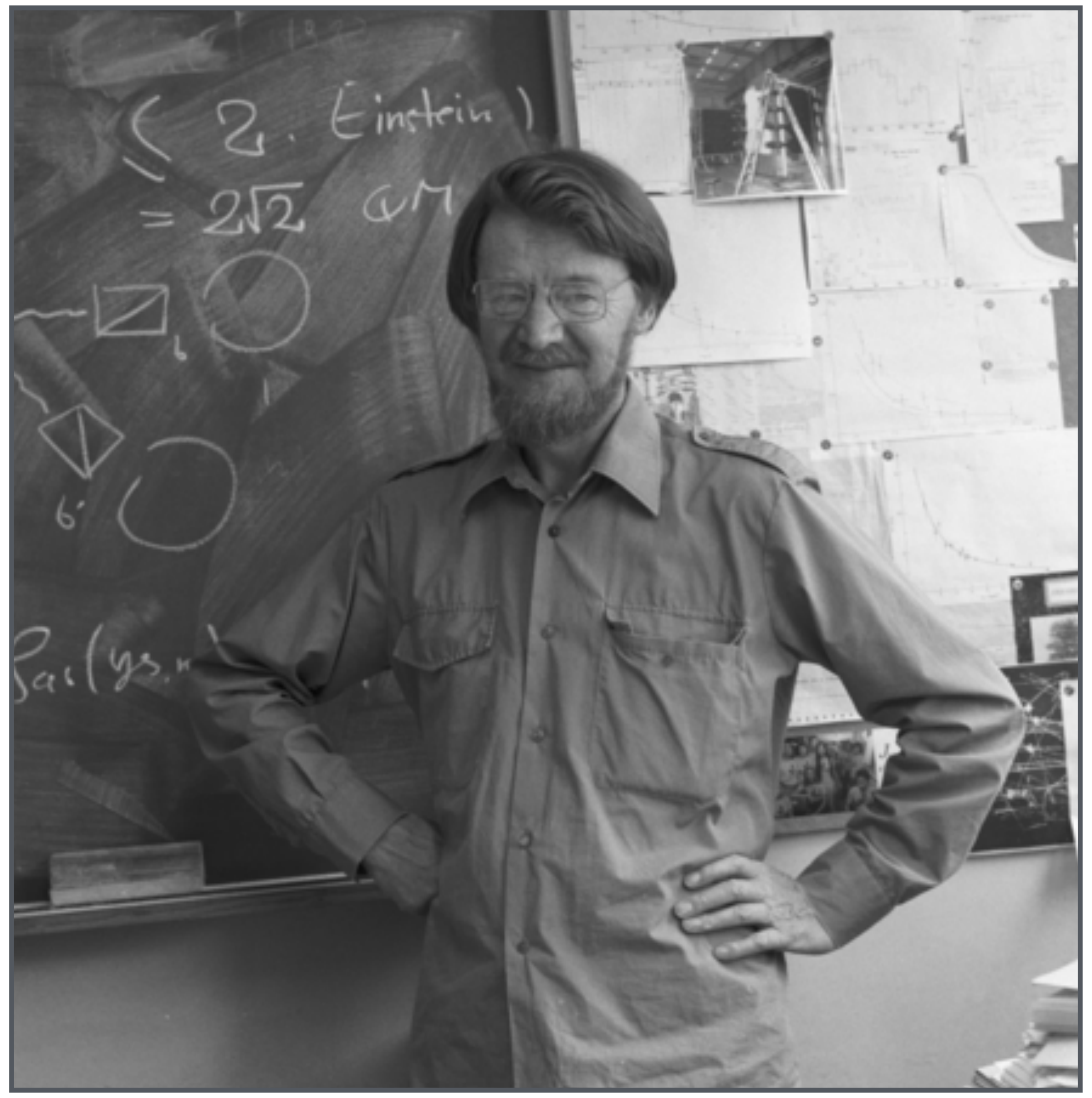

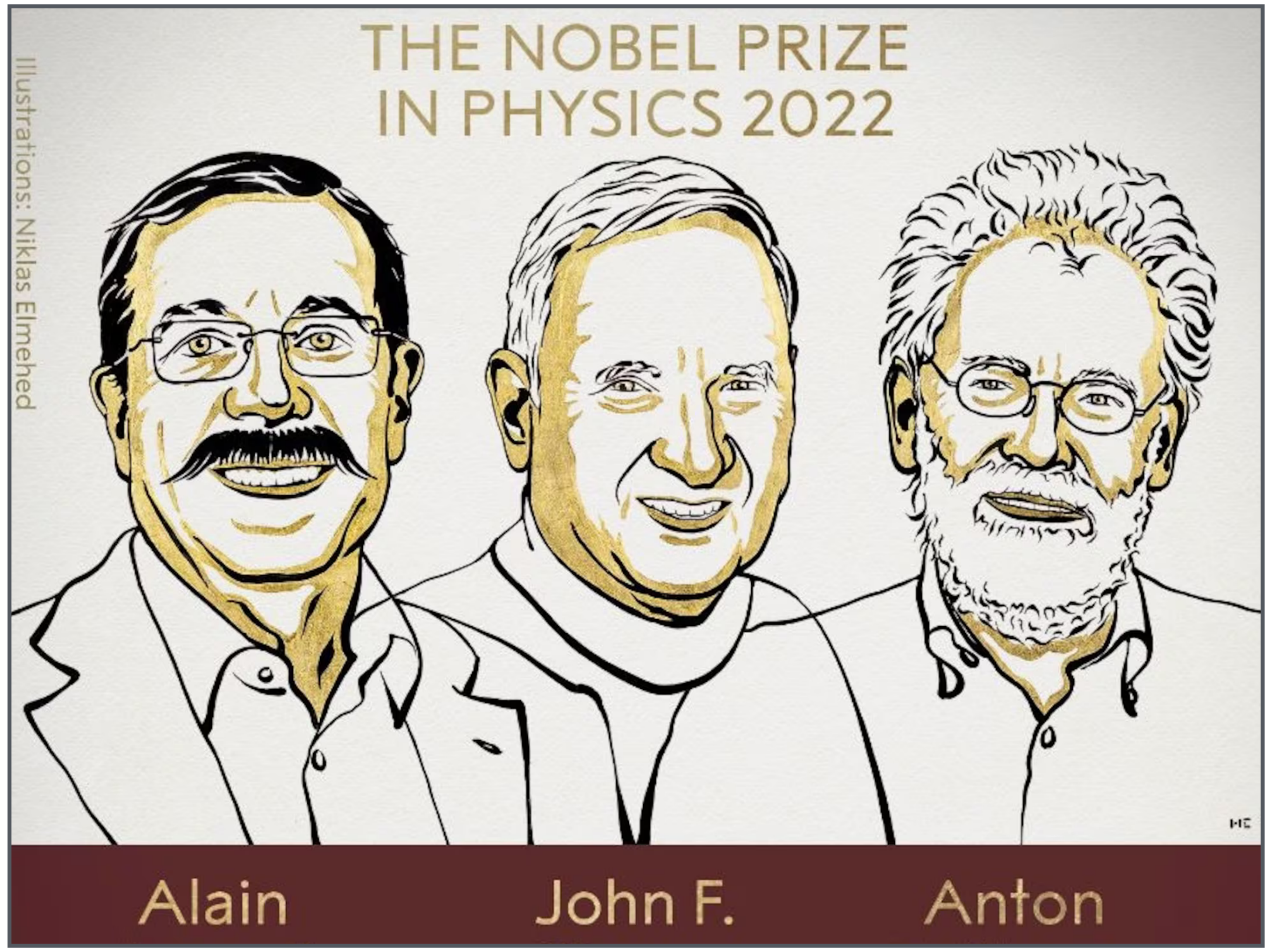

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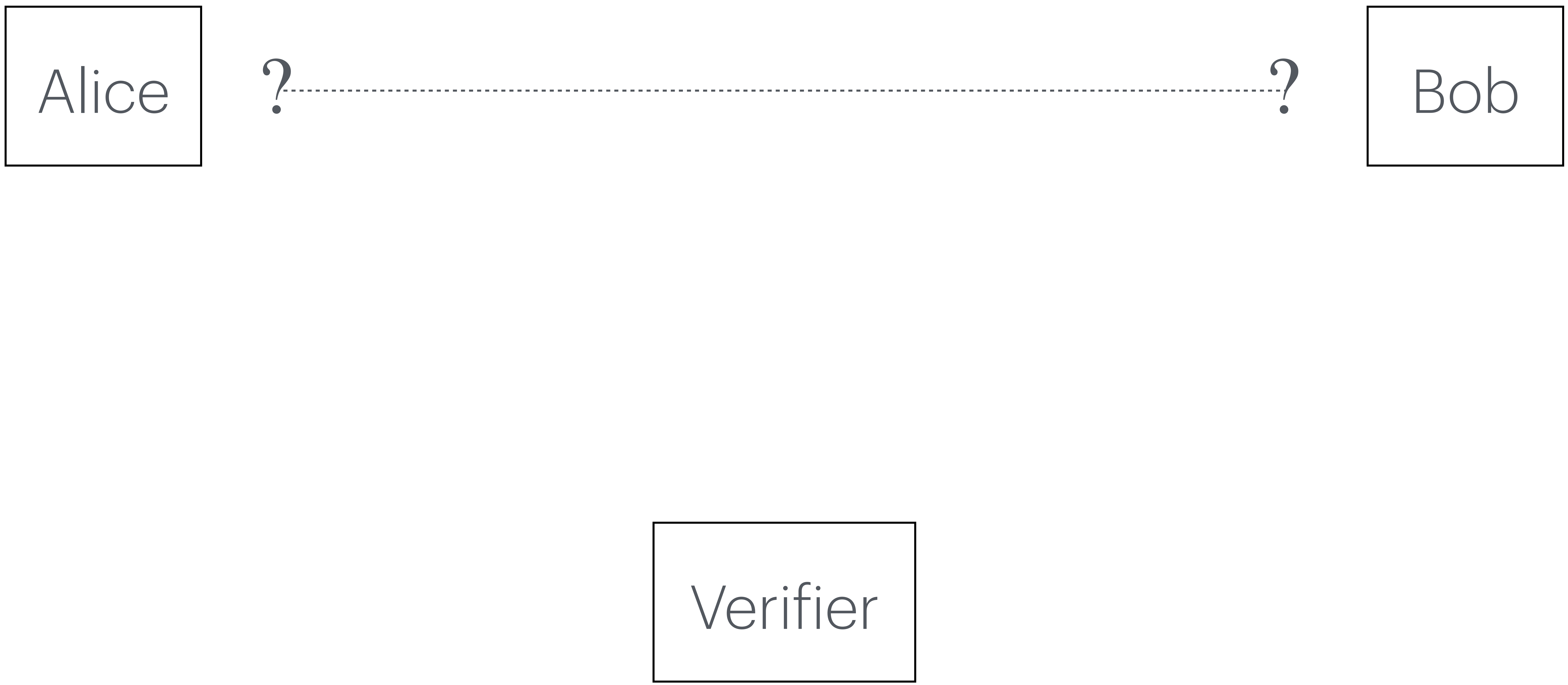

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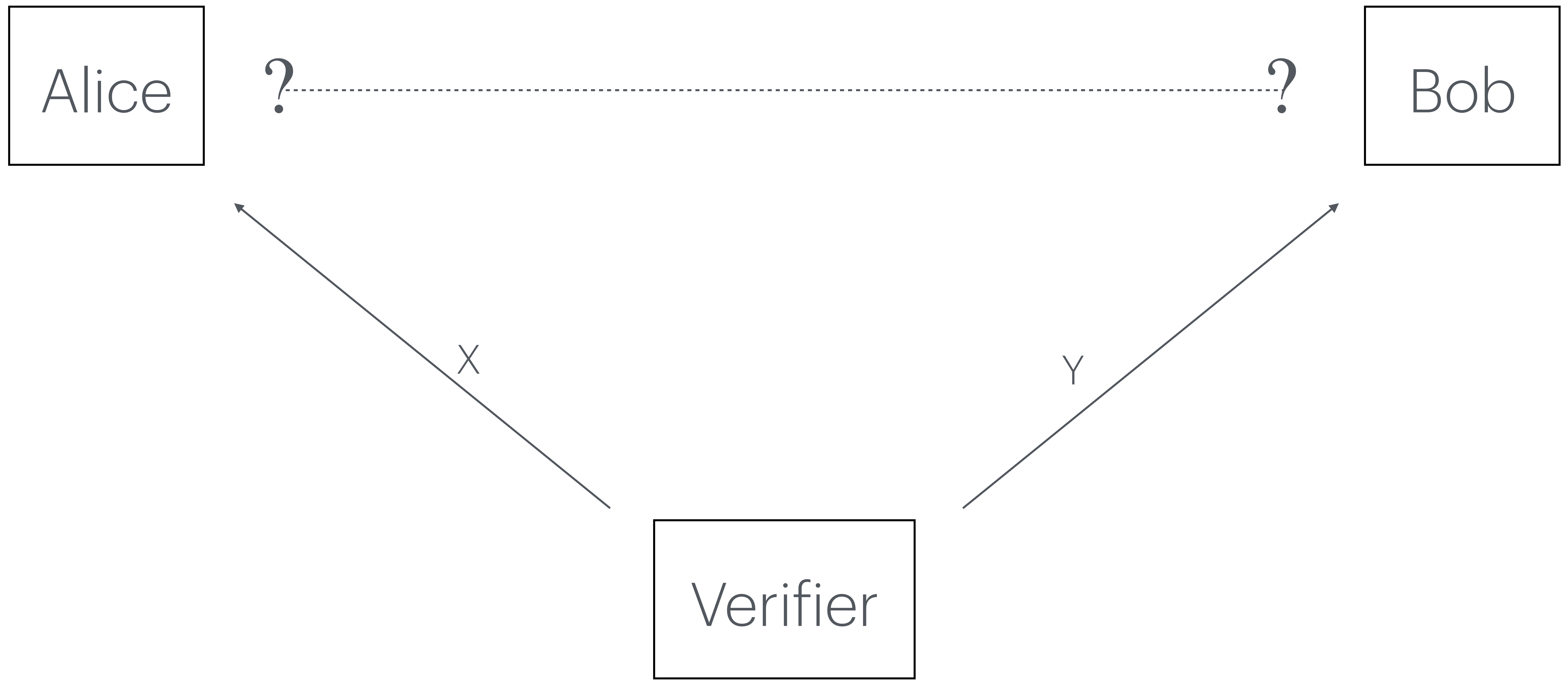

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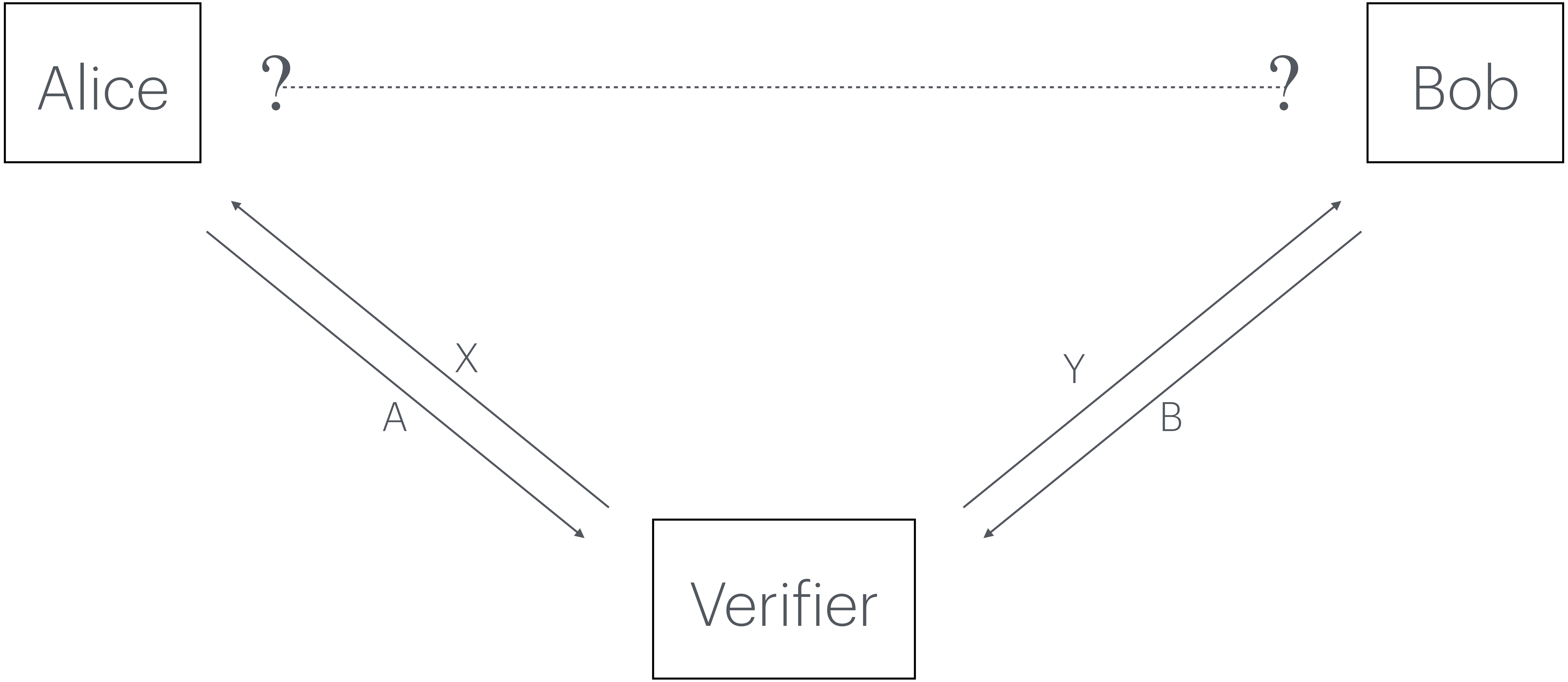

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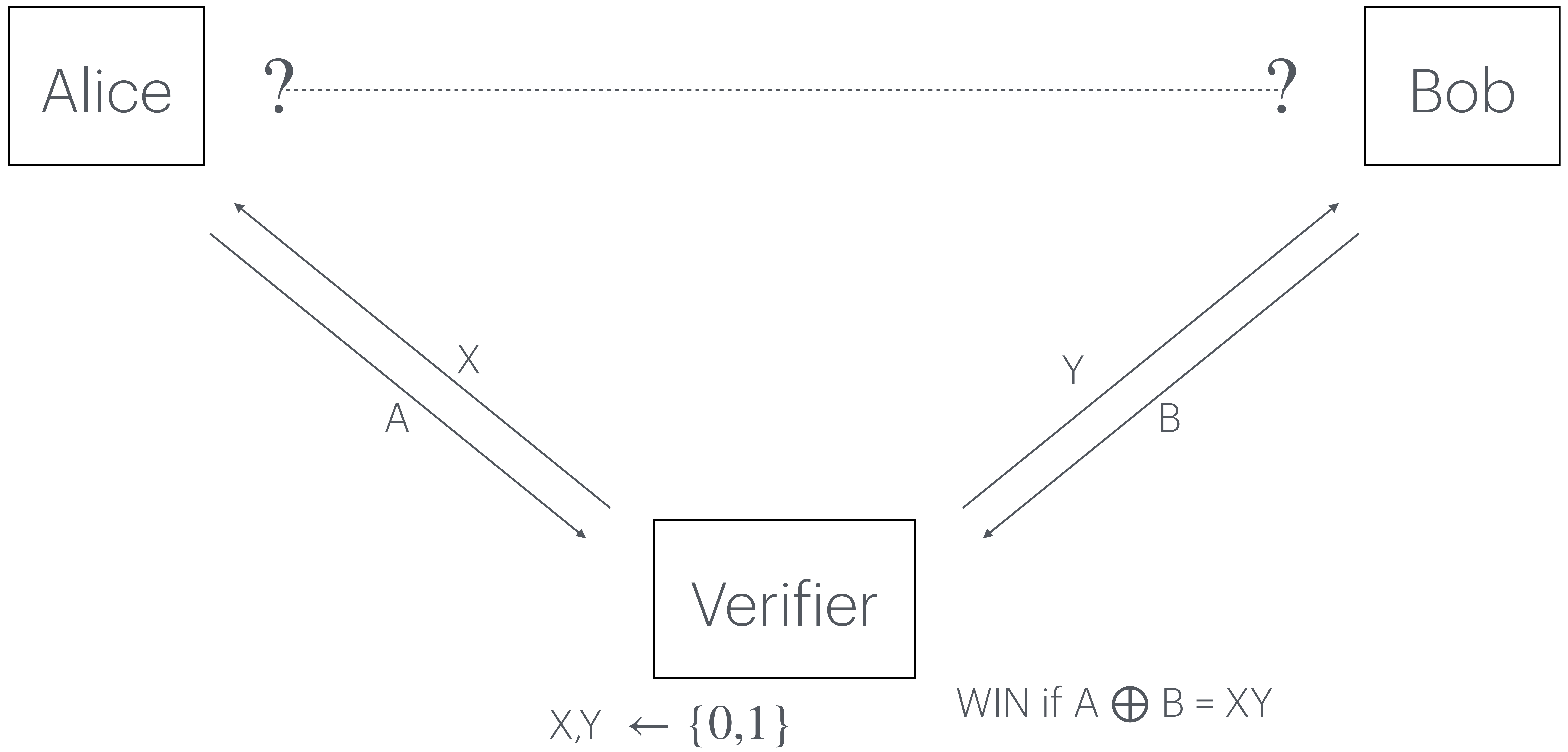

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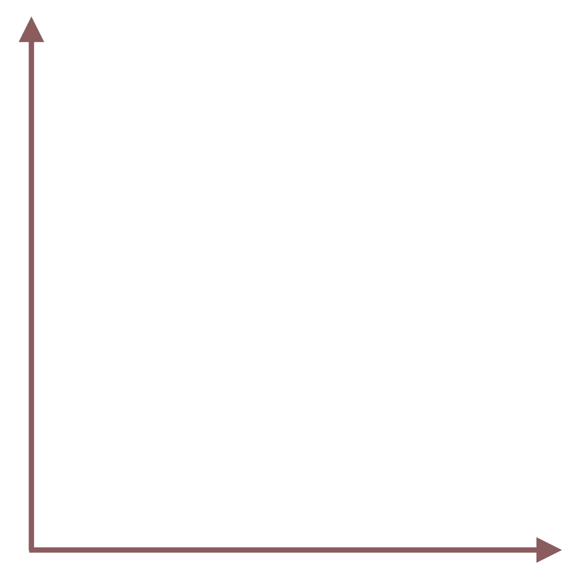

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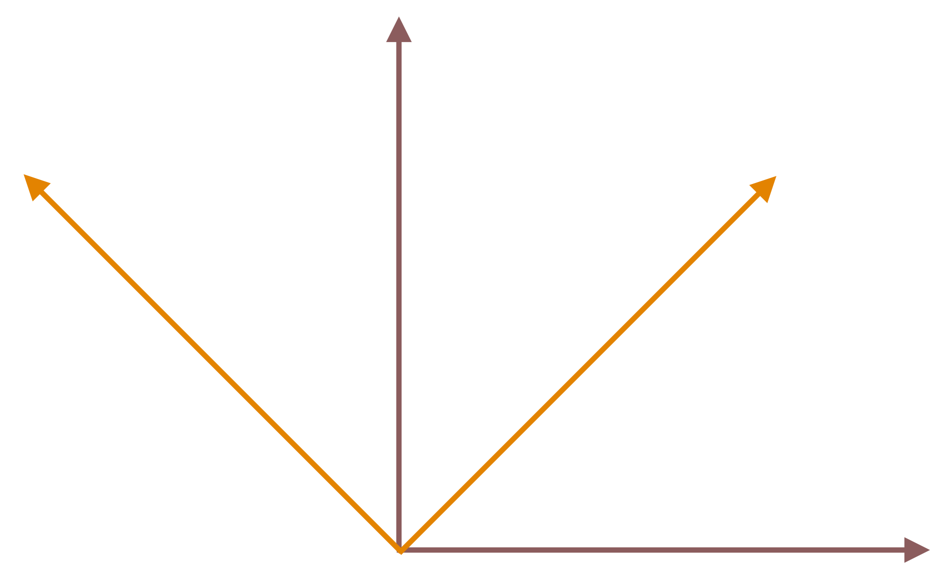

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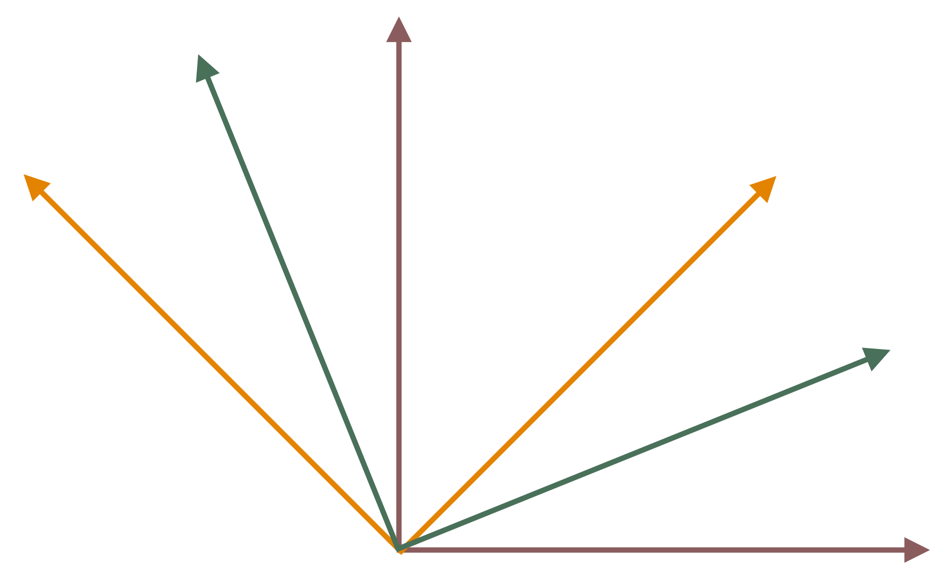

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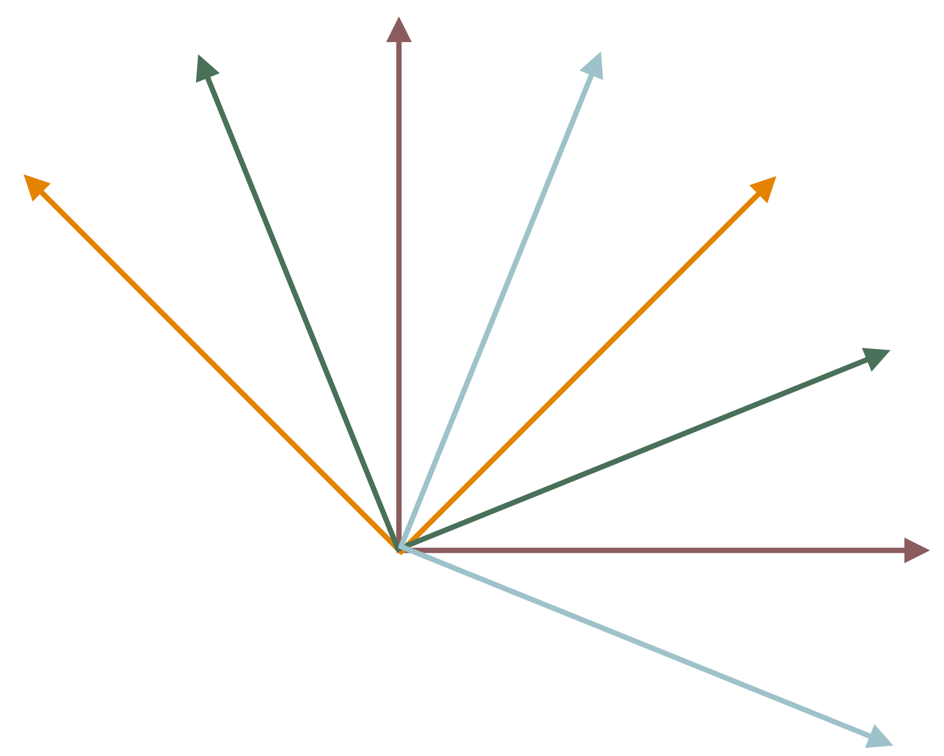

Thank you!