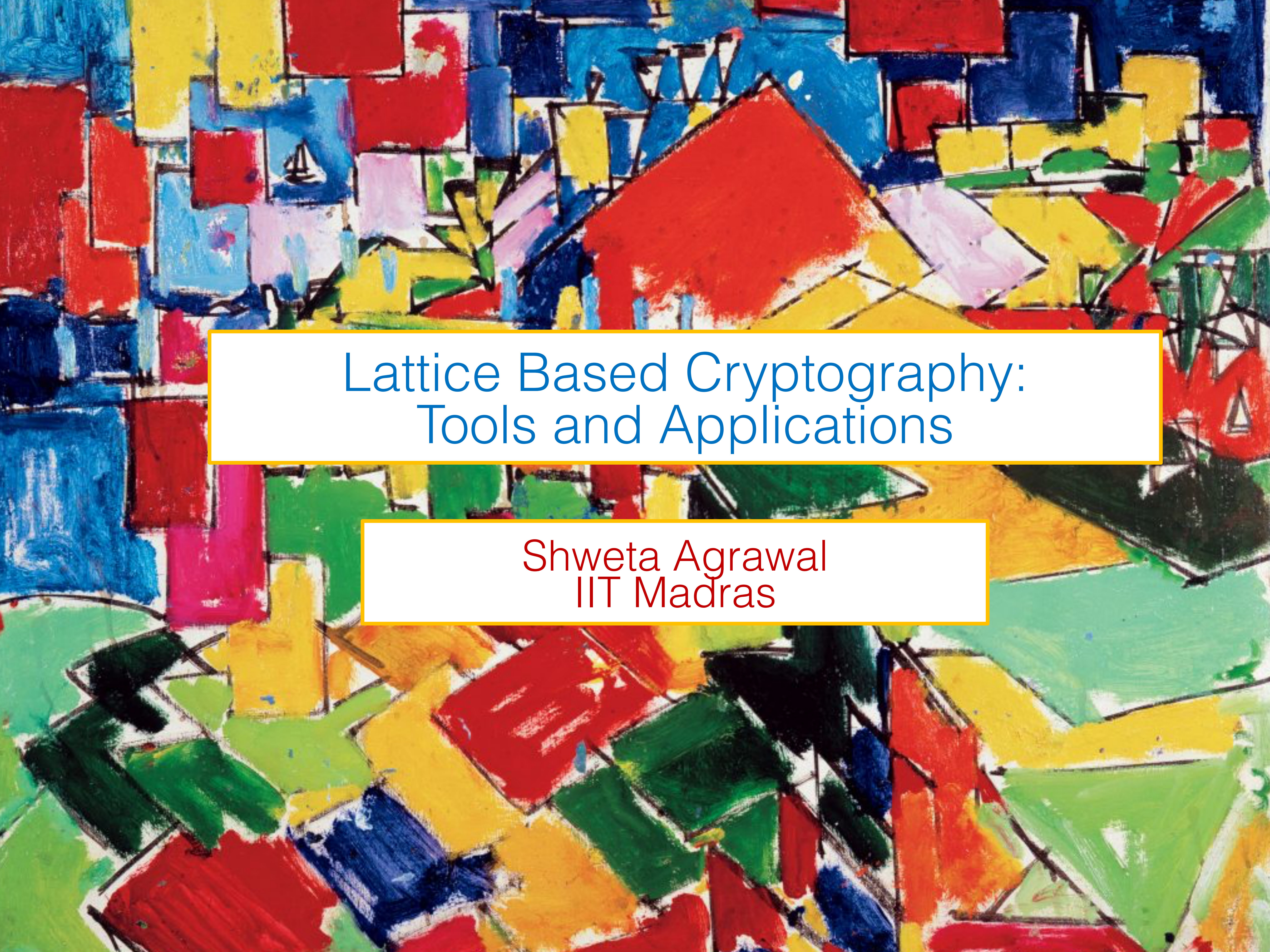

Lattice Based Cryptography: Tools and Applications

Shweta Agrawal
IIT Madras

Computing on Encrypted Data
Personalised Medicine

“The dream for tomorrow’s medicine is to understand the links between DNA and disease — and to tailor therapies accordingly. But scientists have a problem: how to keep genetic data and medical records secure while still enabling the massive, cloud-based analyses needed to make meaningful associations.”

Check Hayden, E. (2015). *Nature*, *519*, 400-401.

“You don’t look anything like the long haired, skinny kid I married 25 years ago. I need a DNA sample to make sure it’s still you.”

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Doesn't FHE solve exactly this?

Access Control on Encrypted Data

Prof. Bob wants to store encrypted file so that:

- Other Professors or admin assistants of CS group can open it
- Encrypt file for each of them?
- If someone quits or new person joins? Re-encrypt ?
- Organizational nightmare !

Access Control on Encrypted Data

Prof. Bob wants to store encrypted file so that:

Access Control on Encrypted Data

Prof. Bob wants to store encrypted file so that:

What he really wants:
Encryption for formula

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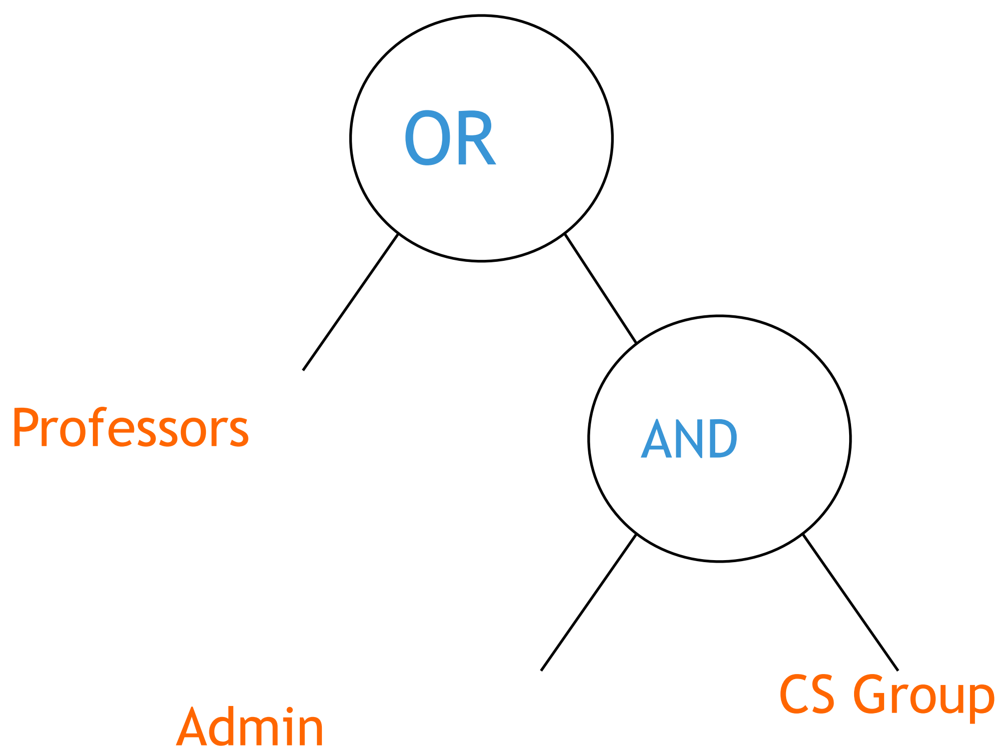

What do we want?

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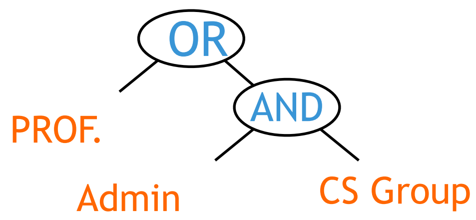

What do we want?

What do we want?

PROF OR {Admin AND CS}

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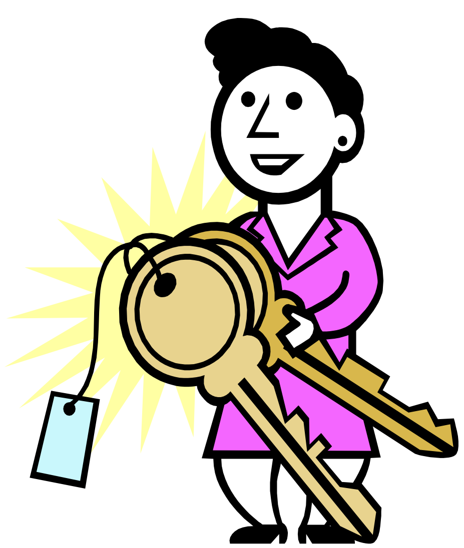

Key Authority

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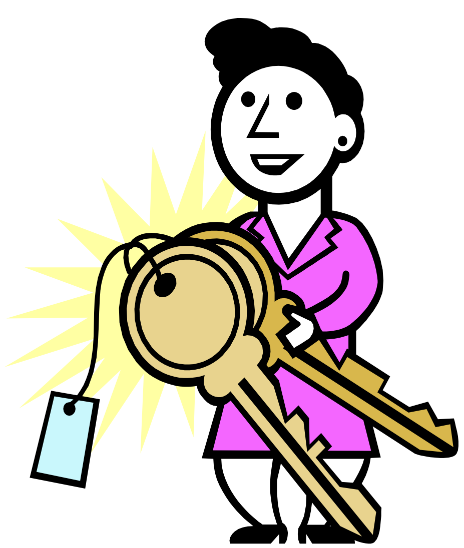

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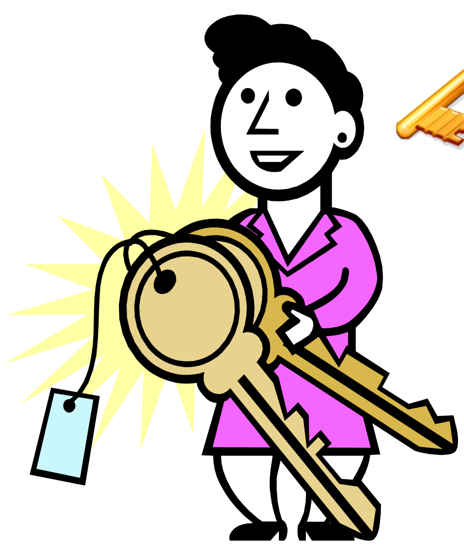

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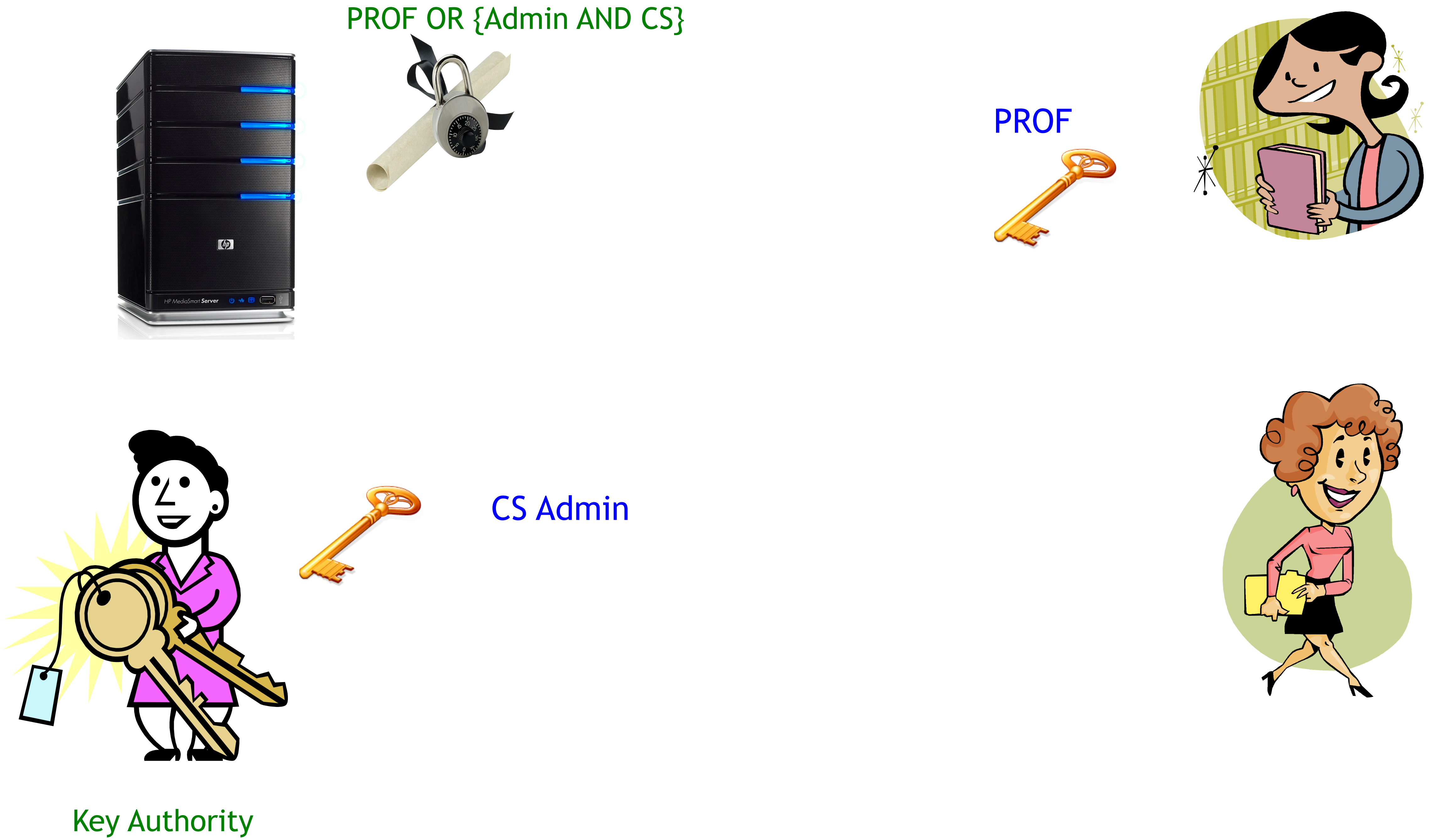

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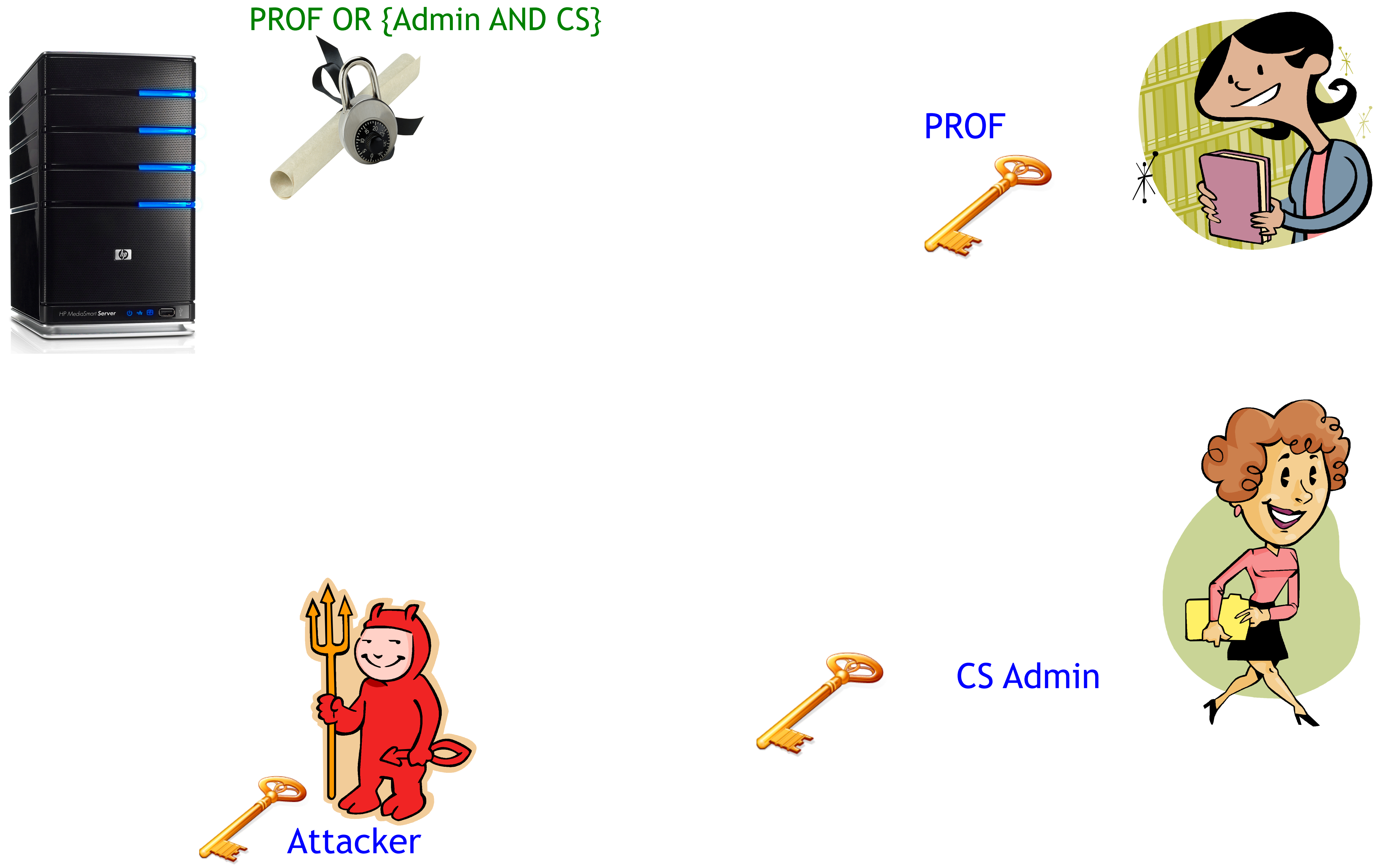

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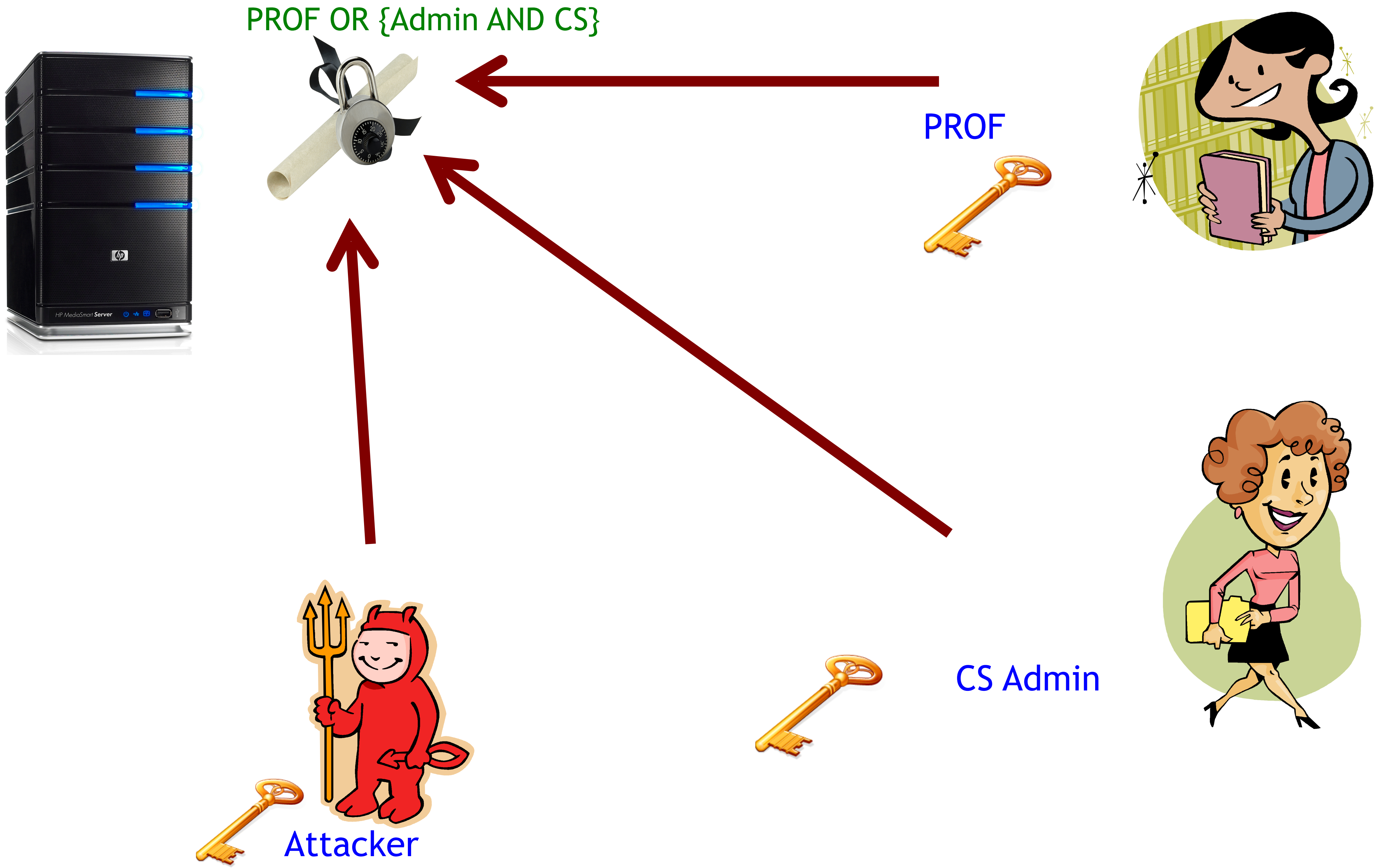

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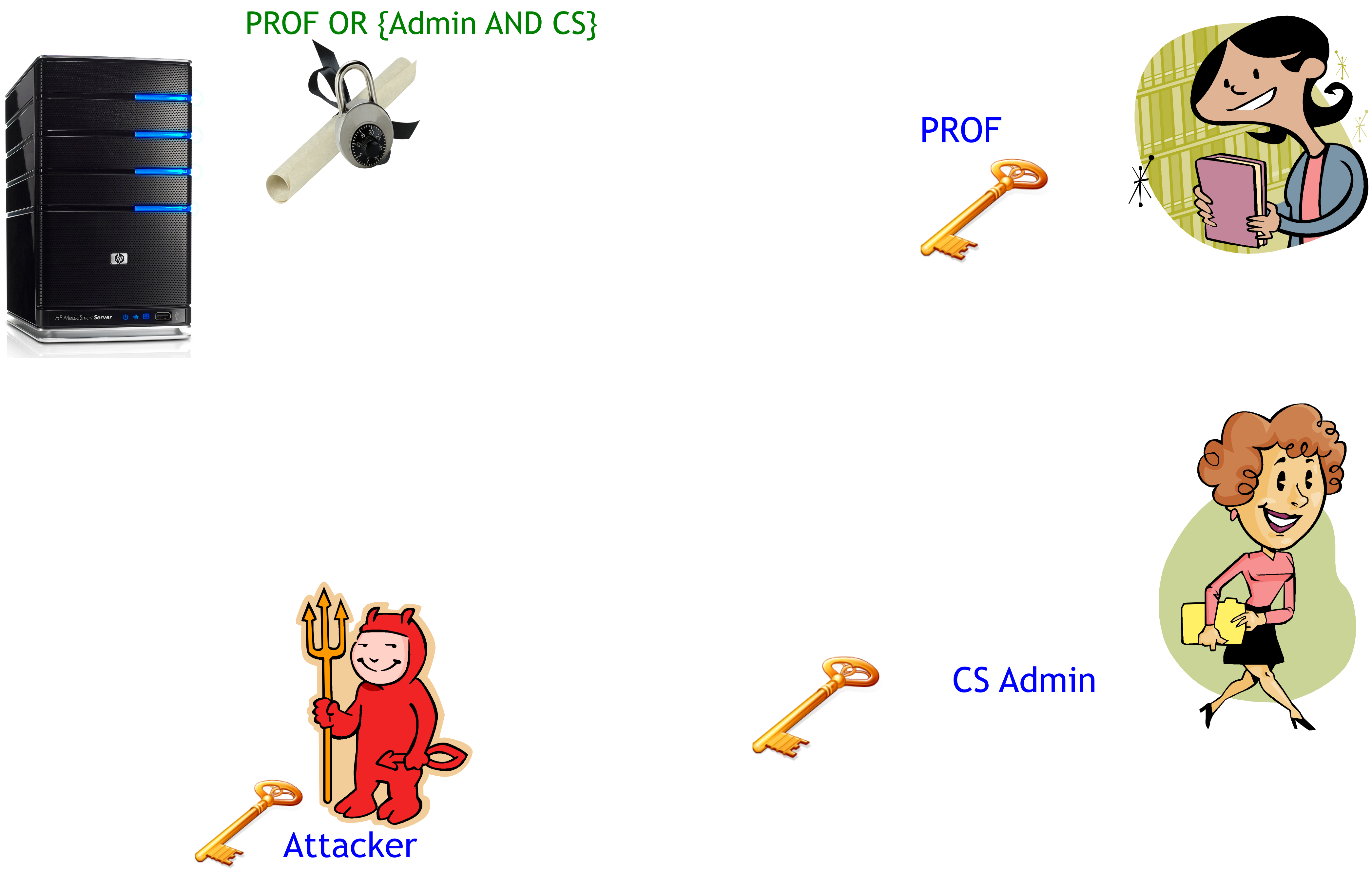

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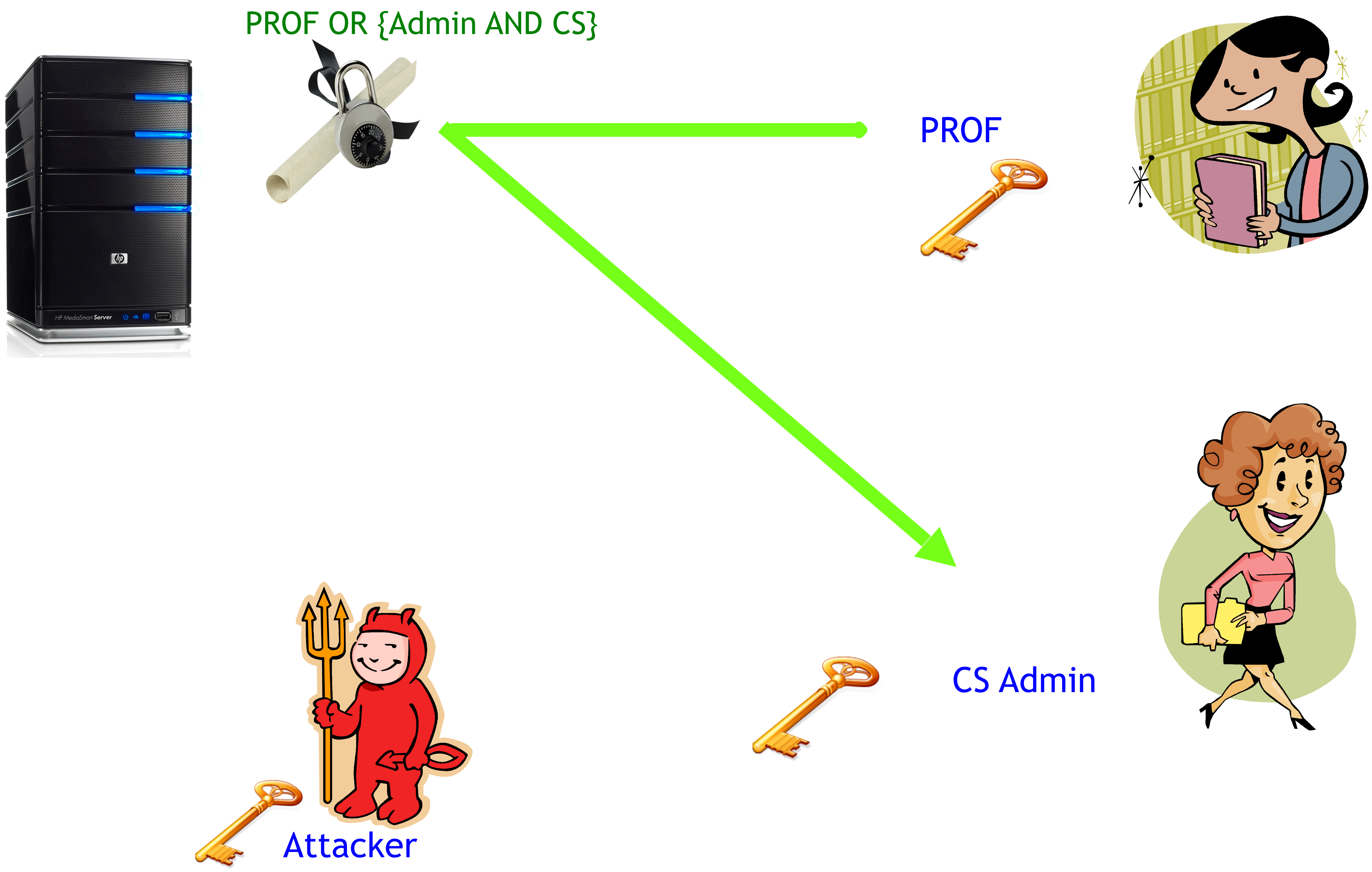

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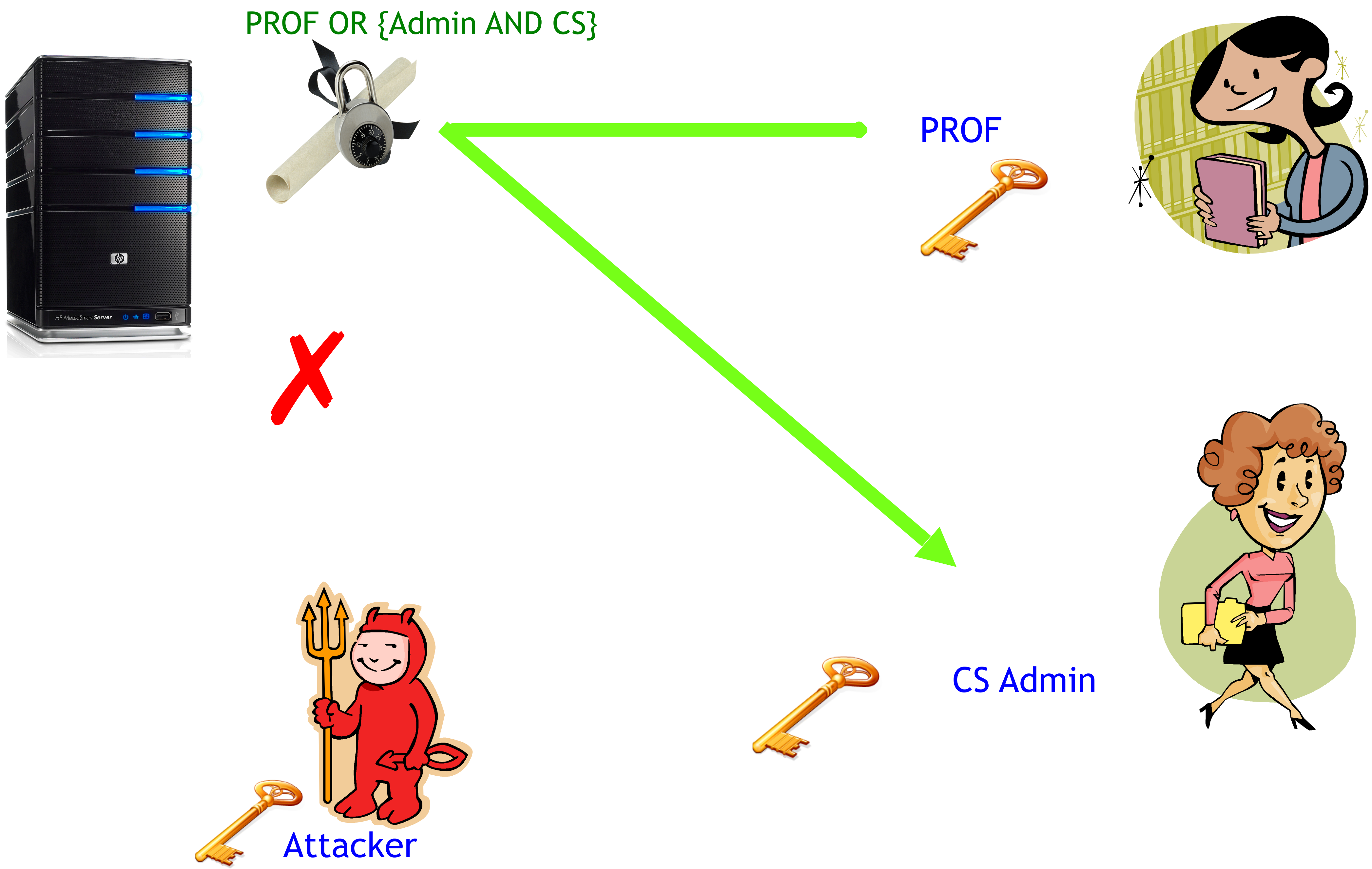


Need New Tools & Techniques!
Main Tool: Lattice Trapdoors

Trapdoor Functions

Trapdoor Functions

Generate (f, T)

Trapdoor Functions

Generate (f, T)

$f : D \to R,$

Trapdoor Functions

Generate (f, T)

$f : D \rightarrow R$, One Way

Trapdoor Functions

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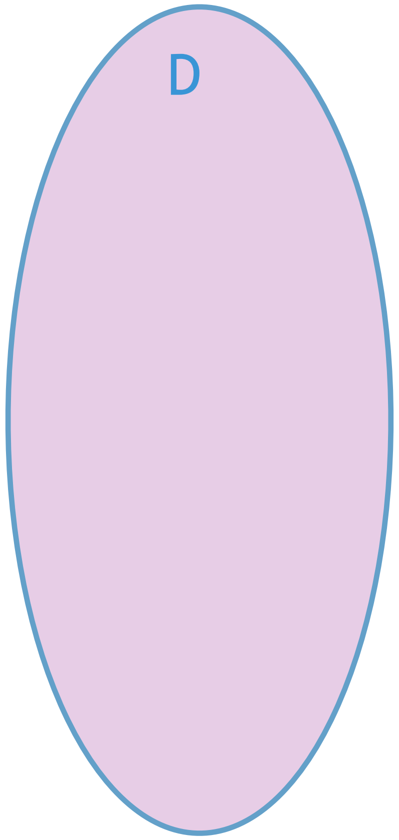

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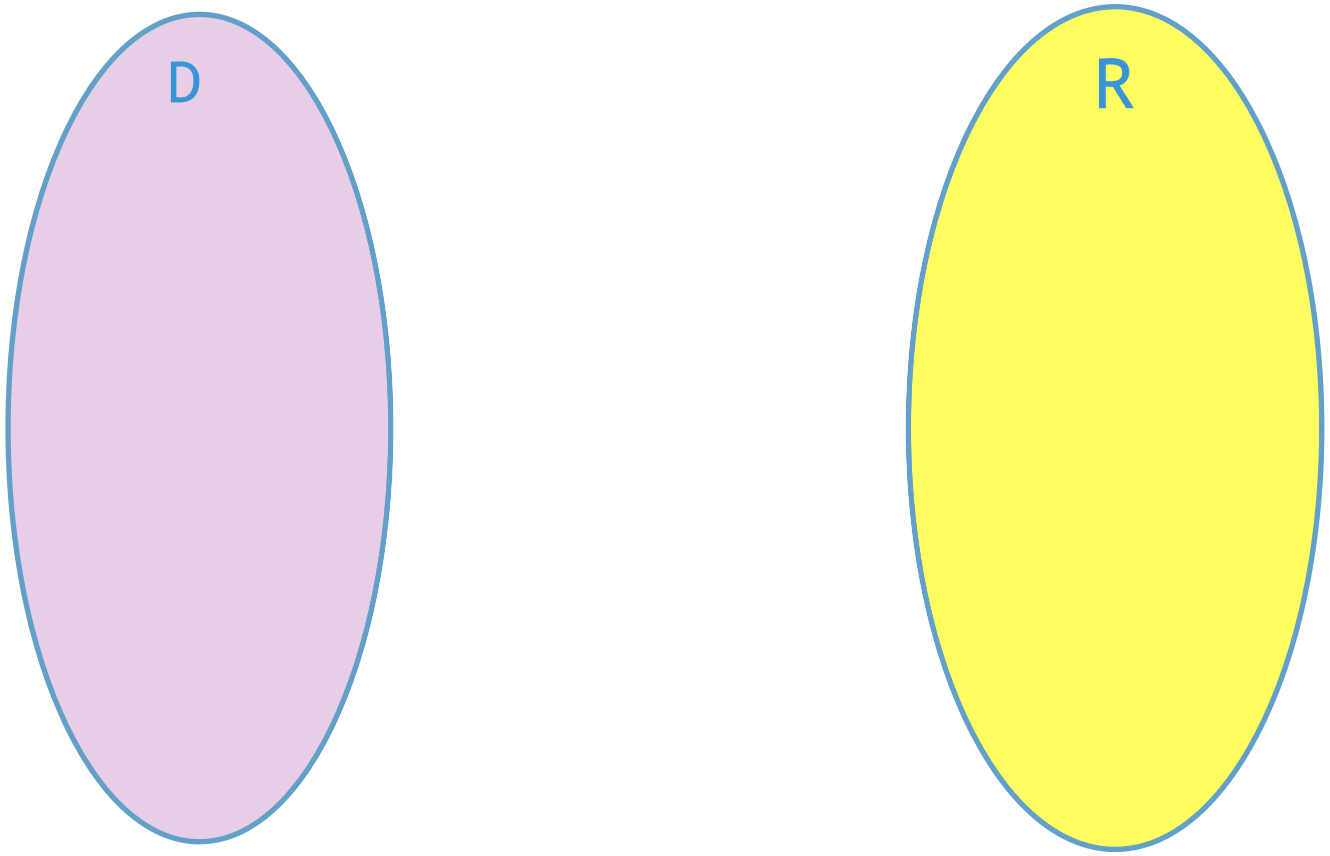

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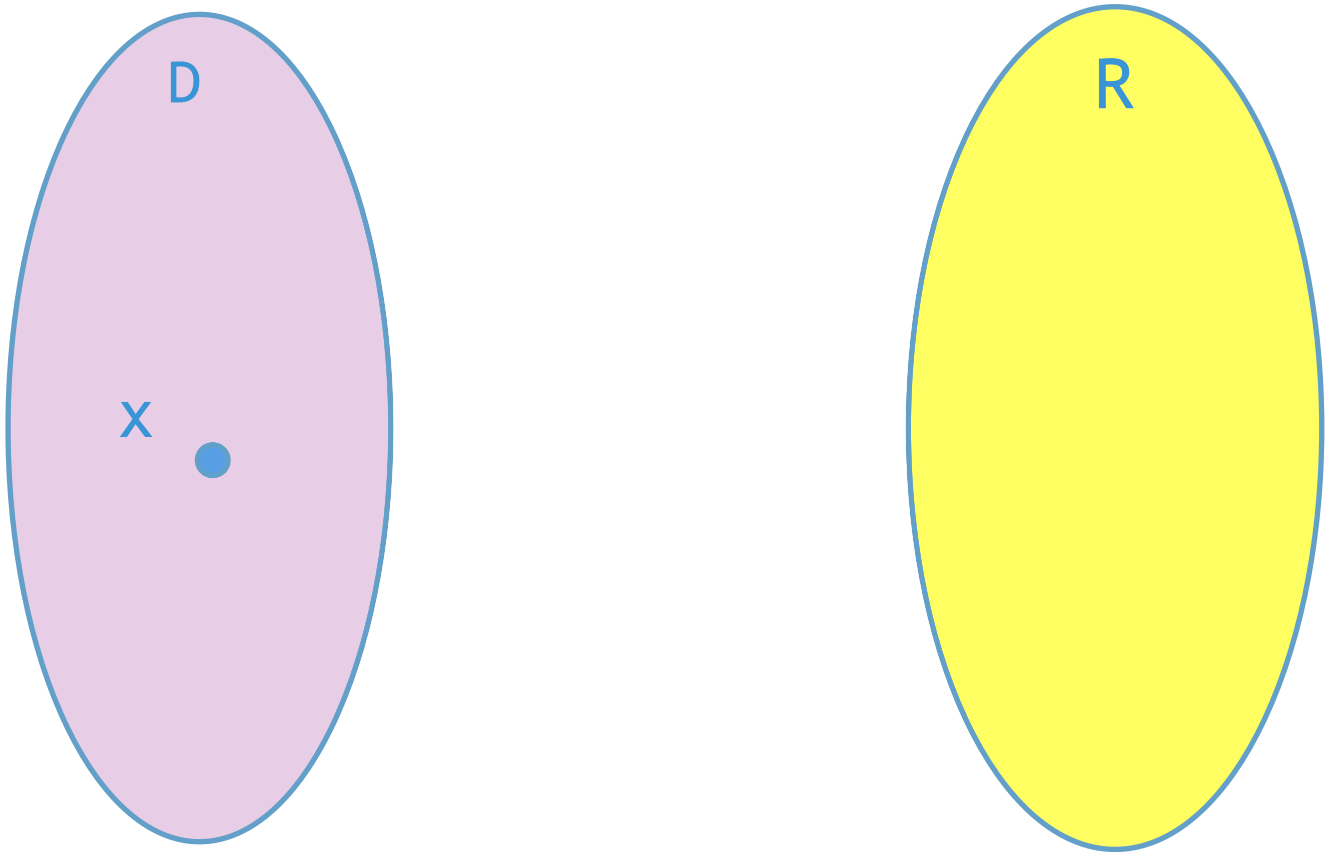

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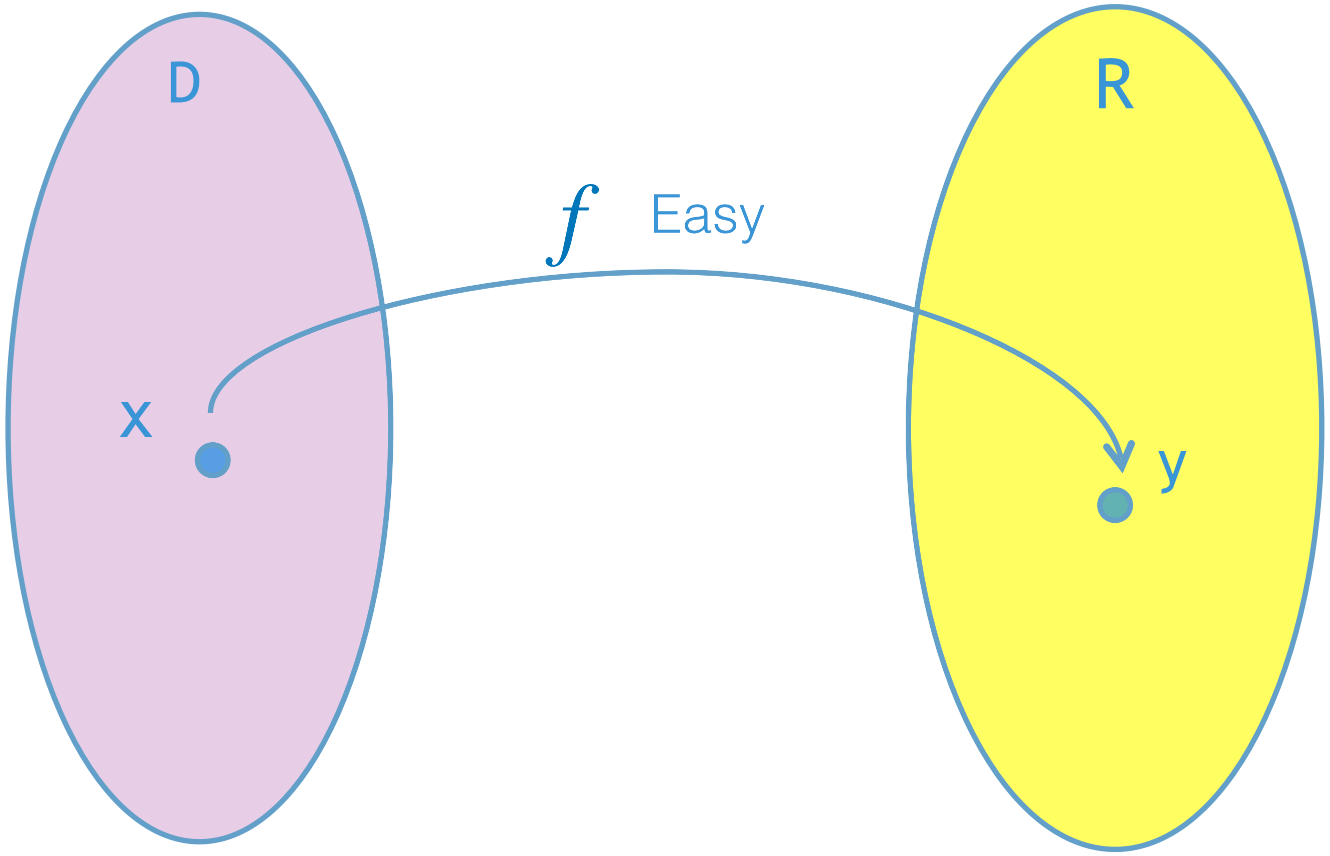

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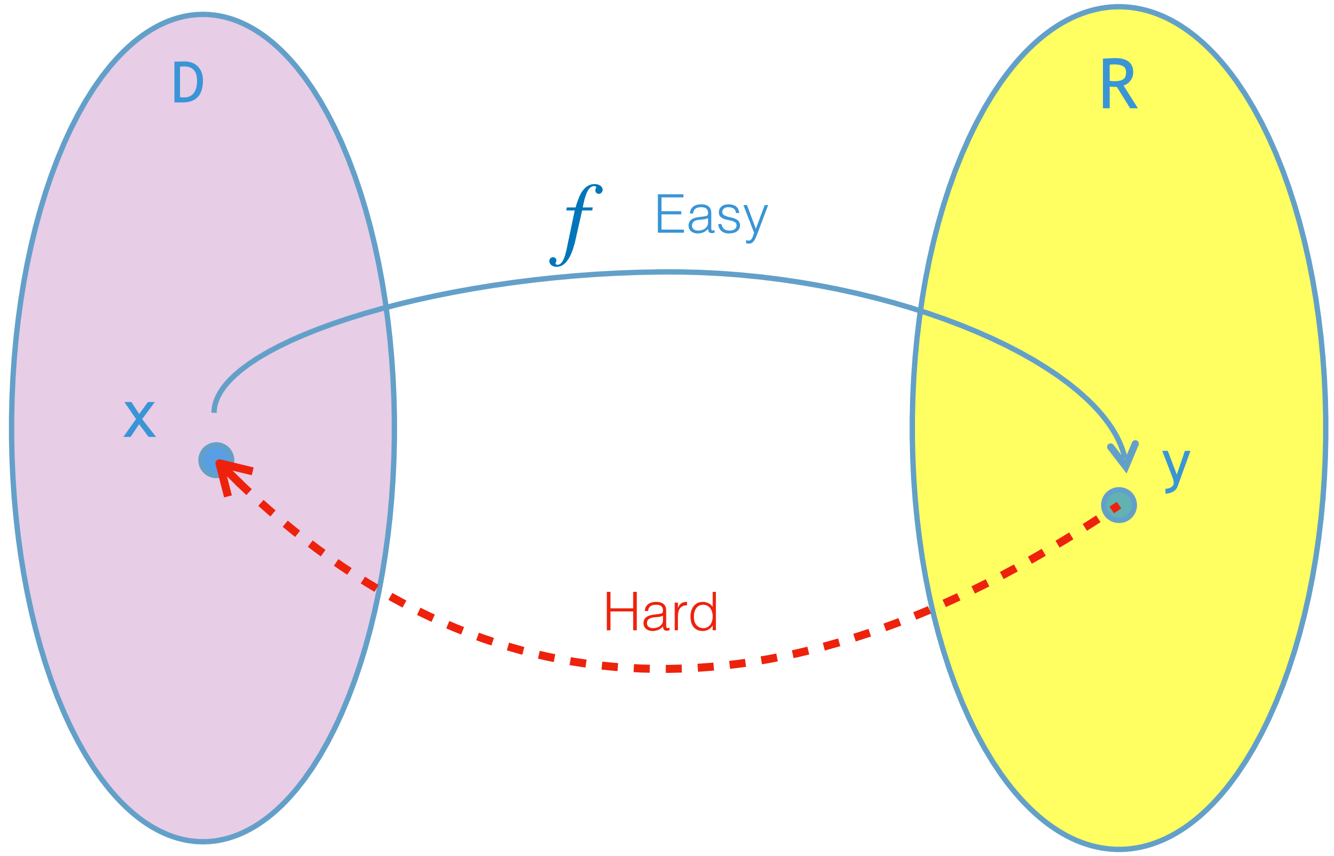

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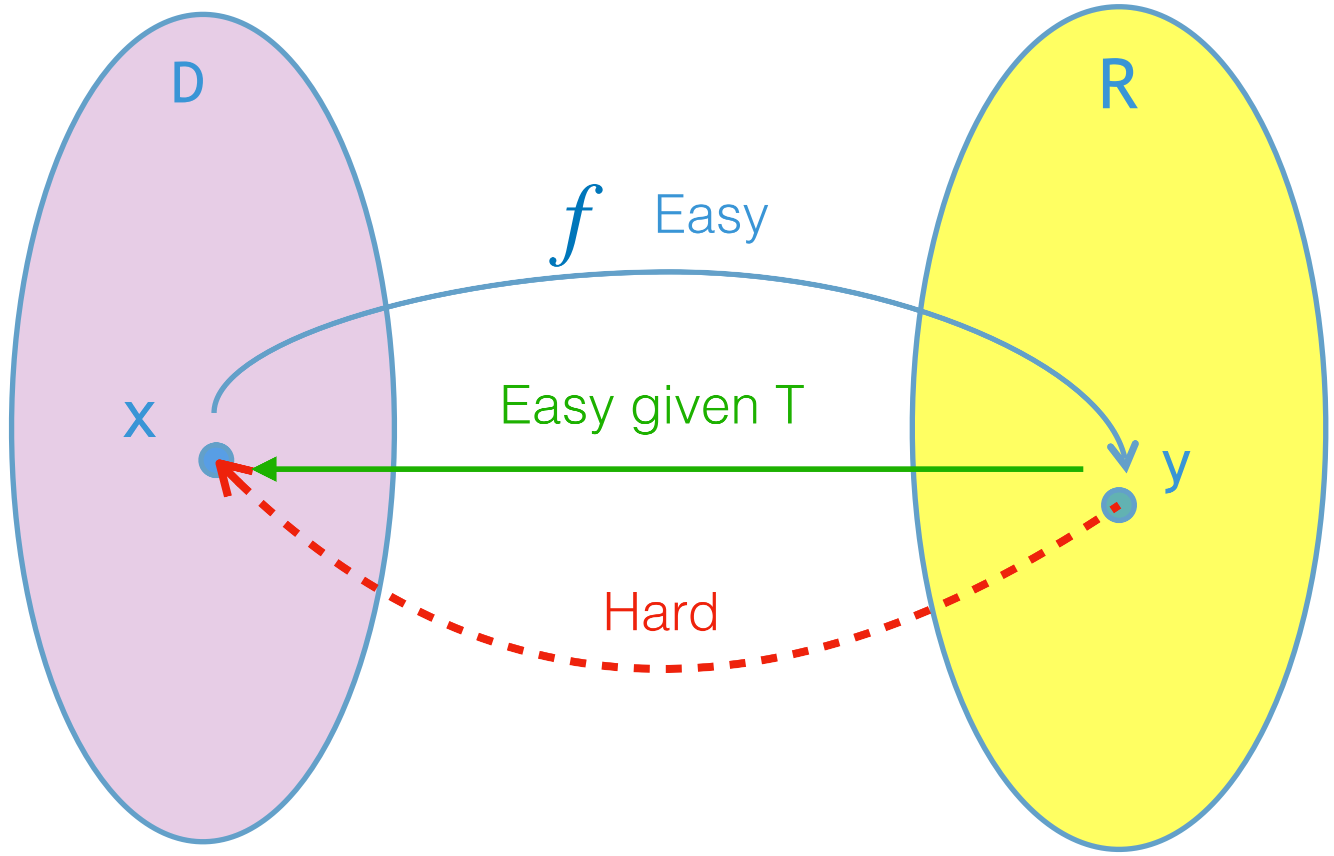
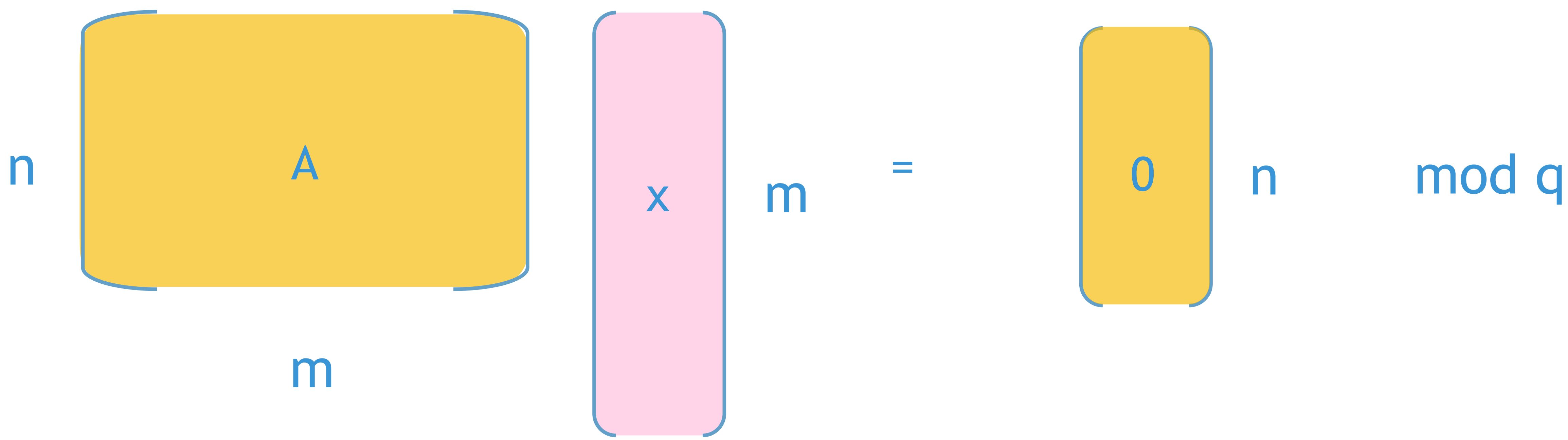


Short Integer Solution Problem

Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, $q = \text{poly}(n)$, $m = \Omega(n \log q)$

Given matrix **A**, find "short" (low norm) vector **x** such that

$$\mathbf{A}\,\mathbf{x} = 0 \mod q \;\in \mathbb{Z}_q^n$$



Learning With Errors Problem

Distinguish "noisy inner products" from uniform

Fix uniform $s \in Z_q^n$

a_1 , $b_1 = \langle a_1, s \rangle + e_1$
a_2 , $b_2 = \langle a_2, s \rangle + e_2$
$\vdots$
a_m , $b_m = \langle a_m, s \rangle + e_m$

vs

a'_1 , b'_1
a'_2 , b'_2
$\vdots$
a'_m , b'_m

a_i uniform $\in Z_q^n$, $e_i \sim \phi \in Z_q$

a_i uniform $\in Z_q^n$, b_i uniform $\in Z_q$

Lattice Based One Way Functions

Public Key $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, $q = \text{poly}(n)$, $m = \Omega(n \log q)$

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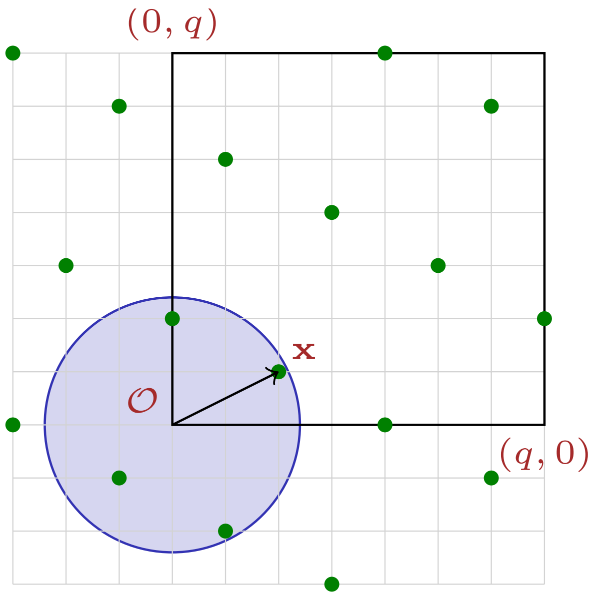
Based on SIS

$$f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} \mod q \ \in \mathbb{Z}_q^n$$

- Short x, surjective
- CRHF if SIS is hard [Ajt96…]



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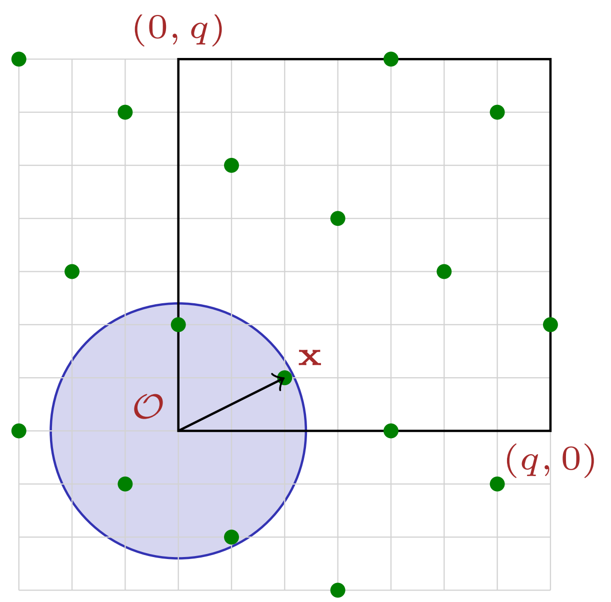


Based on LWE

$$g_{\mathbf{A}}(\mathbf{s}, \mathbf{e}) = \mathbf{s}^t\mathbf{A} + \mathbf{e}^t \mod q \in \mathbb{Z}_q^m$$

- Very short e, injective
- OWF if LWE is hard [Reg05…]

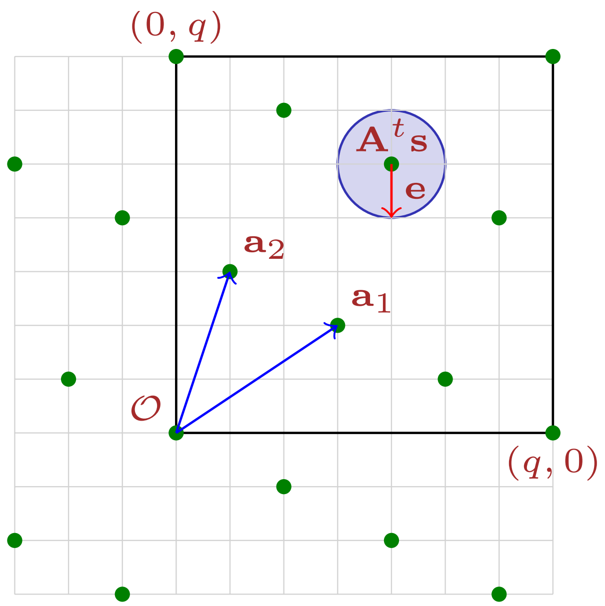


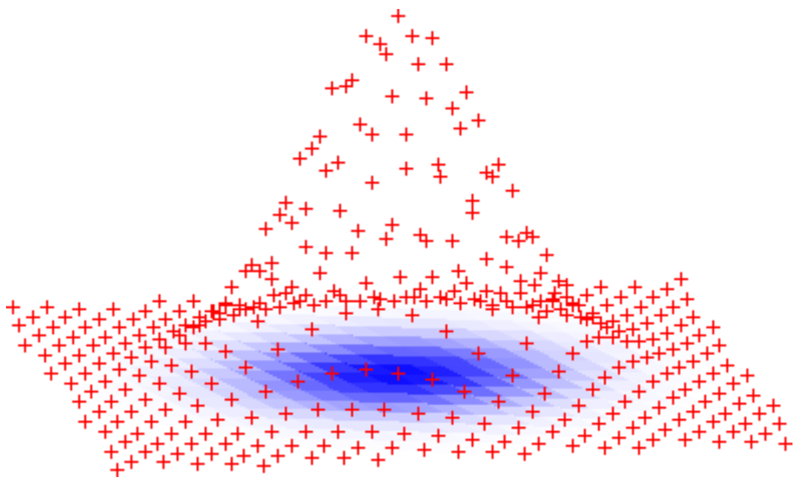
Image Credit: MP12 slides

Inverting functions for Crypto

- Given $\mathbf{u} = f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\ \mathbf{x} \mod q$
- Sample

$$\mathbf{x}' \leftarrow = f_{\mathbf{A}}^{-1}(\mathbf{u})$$

with prob $\propto \exp(-\|\mathbf{x}'\|^2/\sigma^2)$



And

- Given $g_{\mathbf{A}}(\mathbf{s}, \mathbf{e}) = \mathbf{s}^t\mathbf{A} + \mathbf{e}^t \mod q$
- Find unique $(\mathbf{s}, \mathbf{e})$

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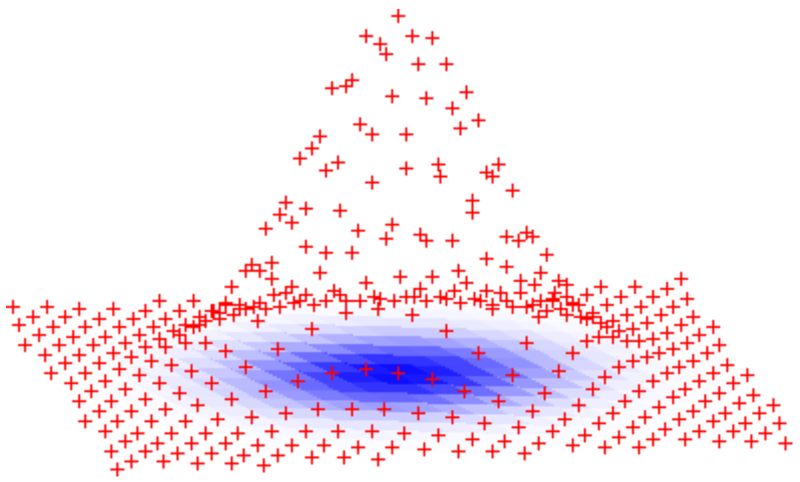

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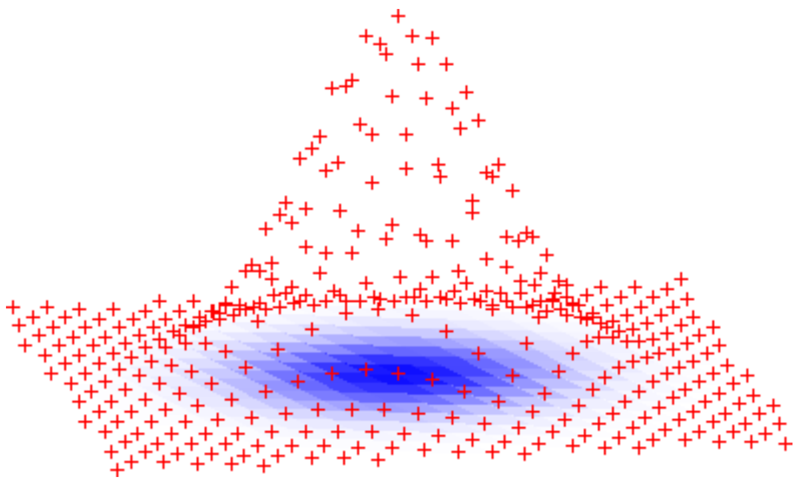

Preimage Sampleable Trapdoor Functions!

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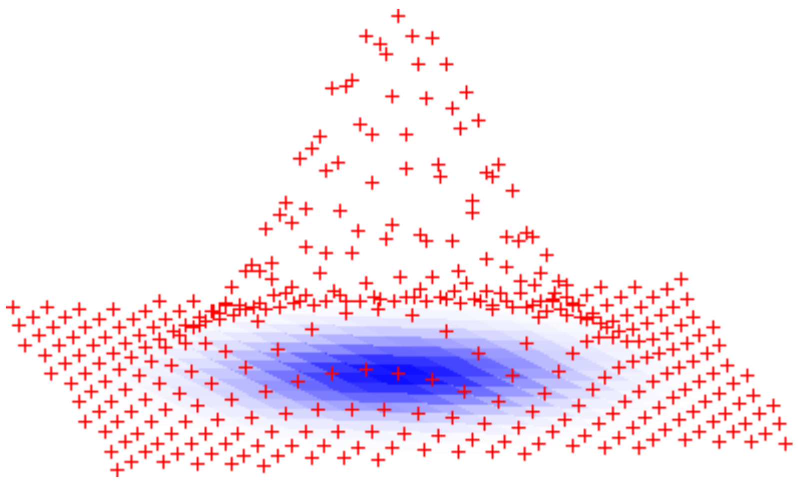

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Generate (x, y) in two equivalent ways

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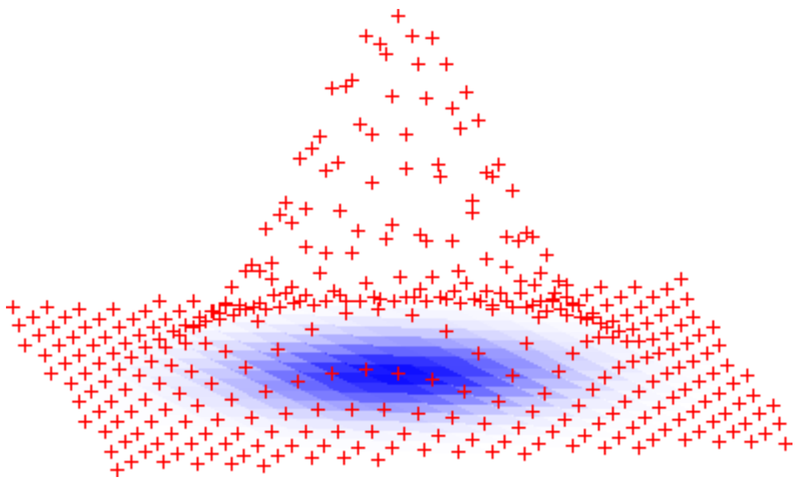

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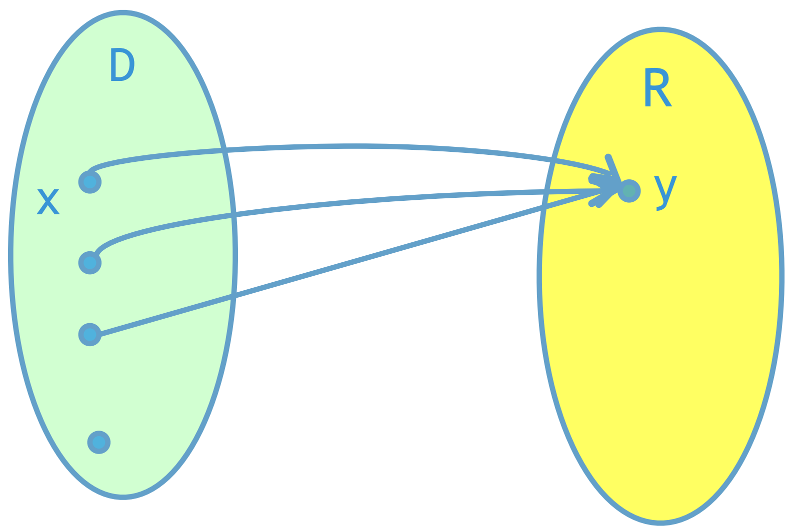

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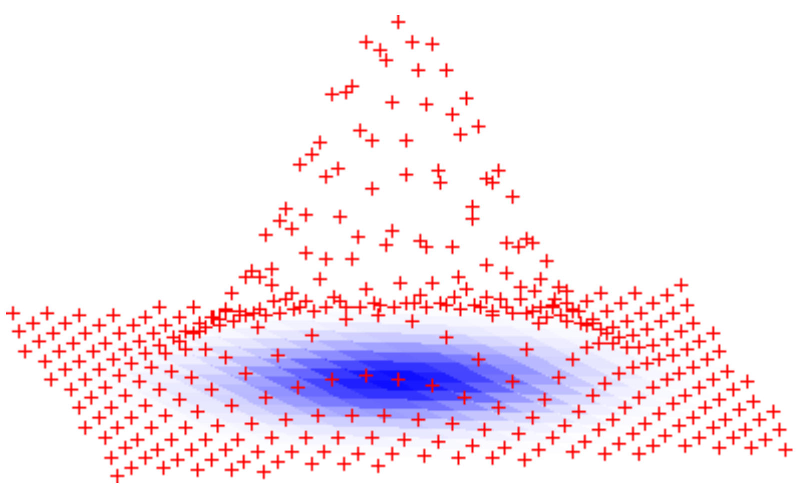

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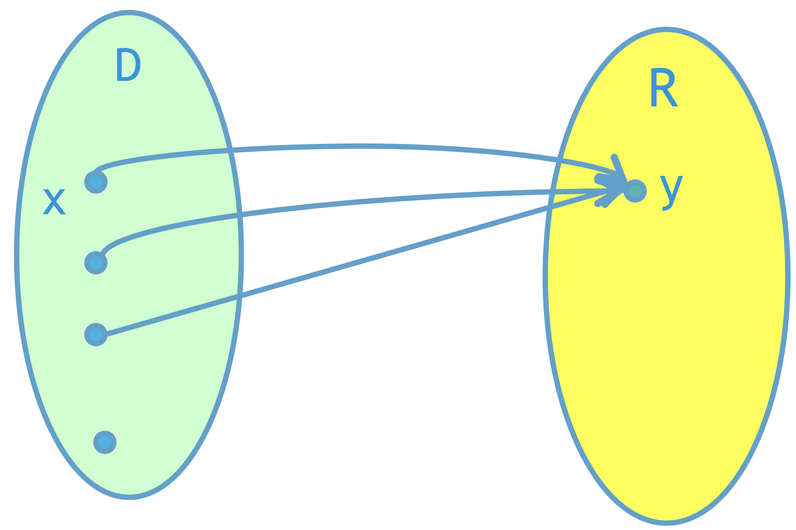

OR

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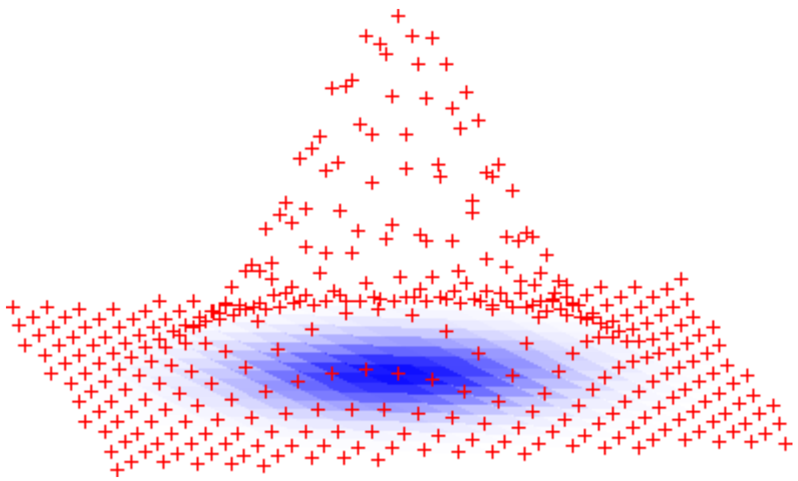

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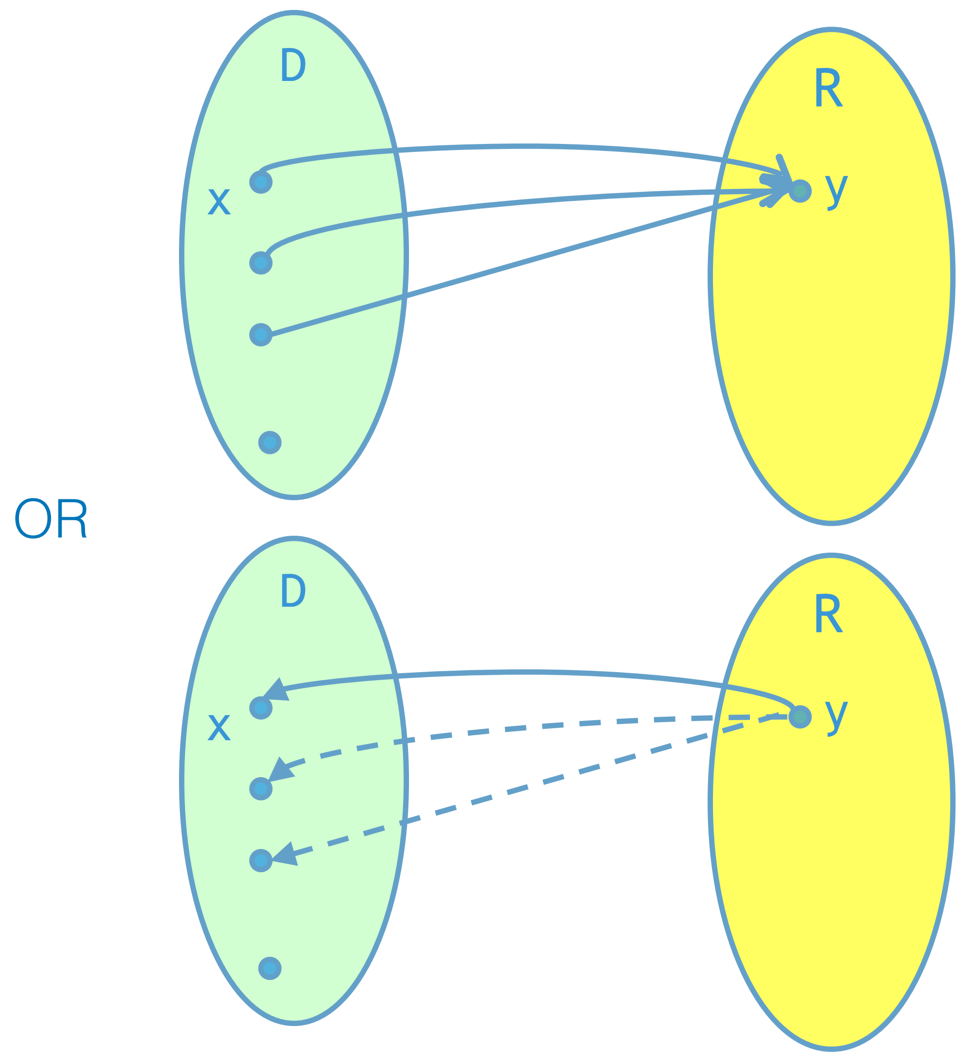

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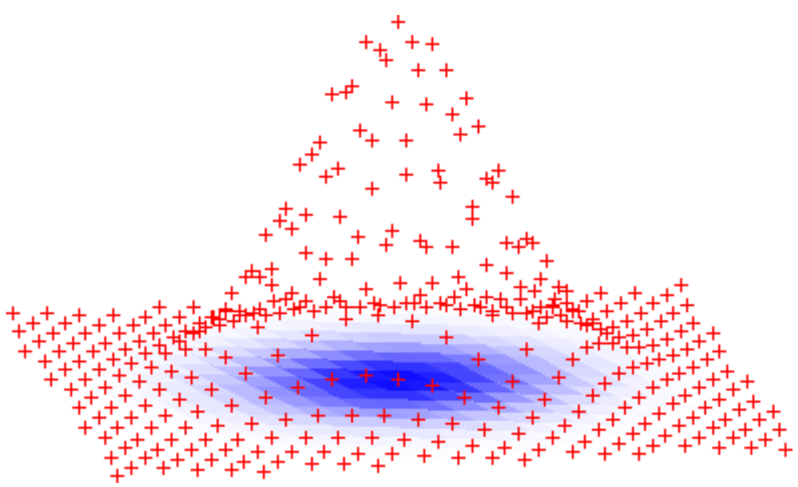

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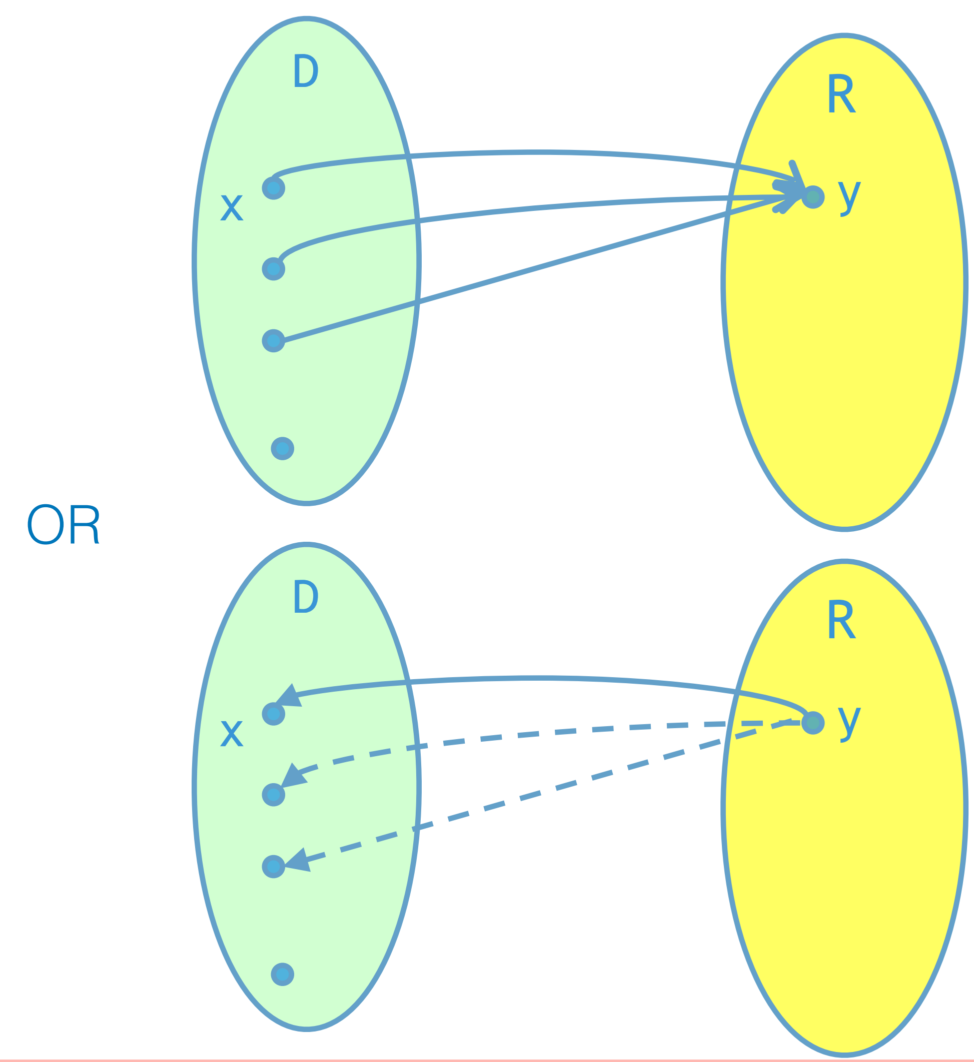

Same Distribution (Discrete Gaussian, Uniform) !

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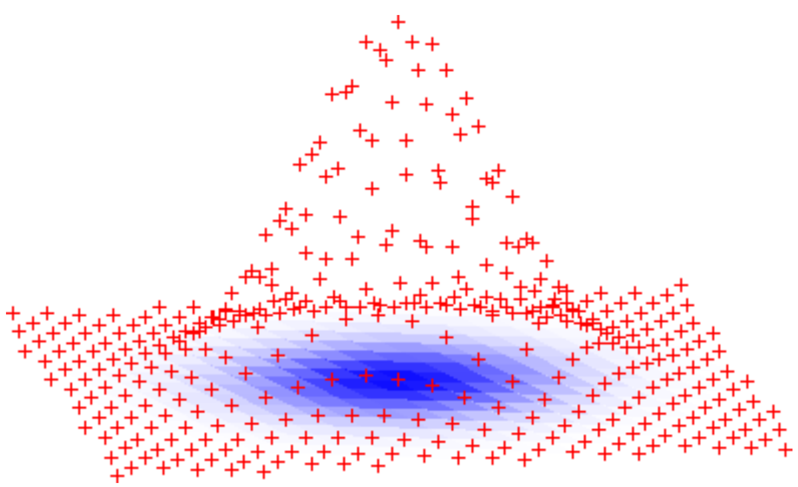
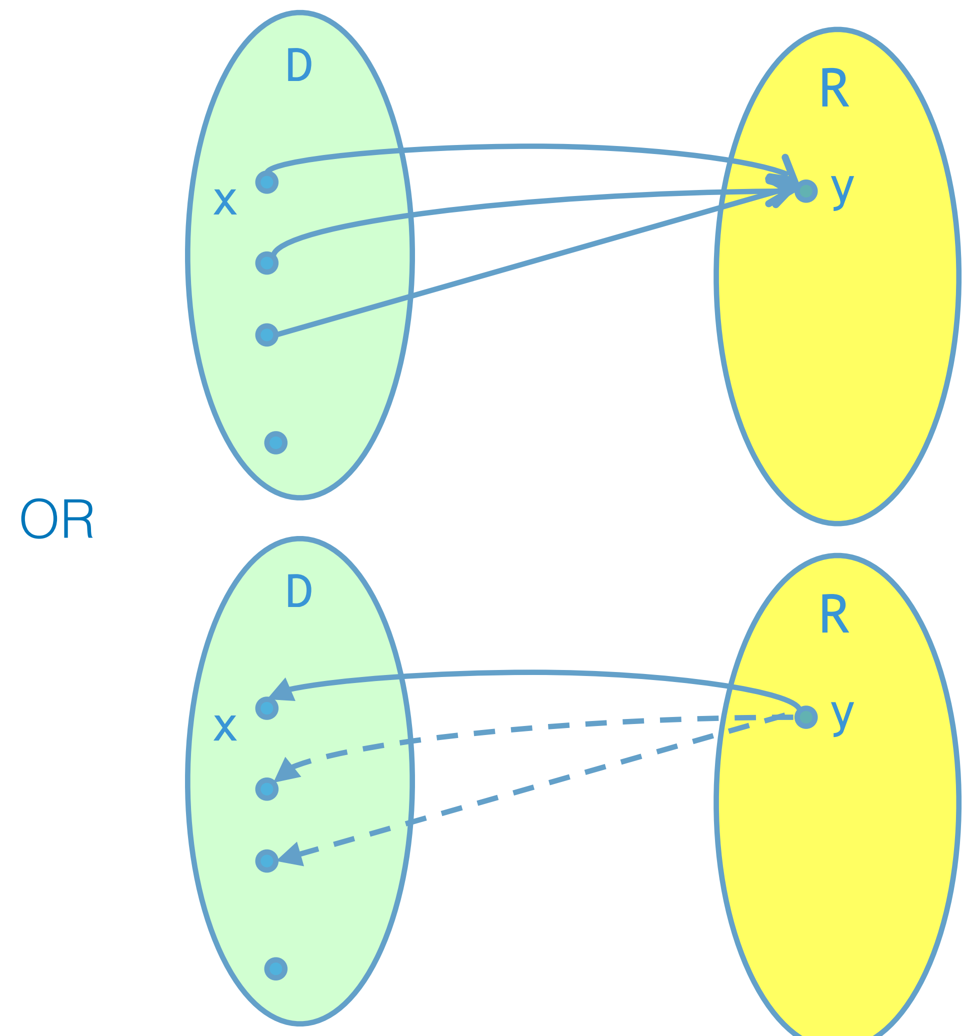


Latter distribution needs lattice trapdoors!

Preimage Sampleable Trapdoor Functions!

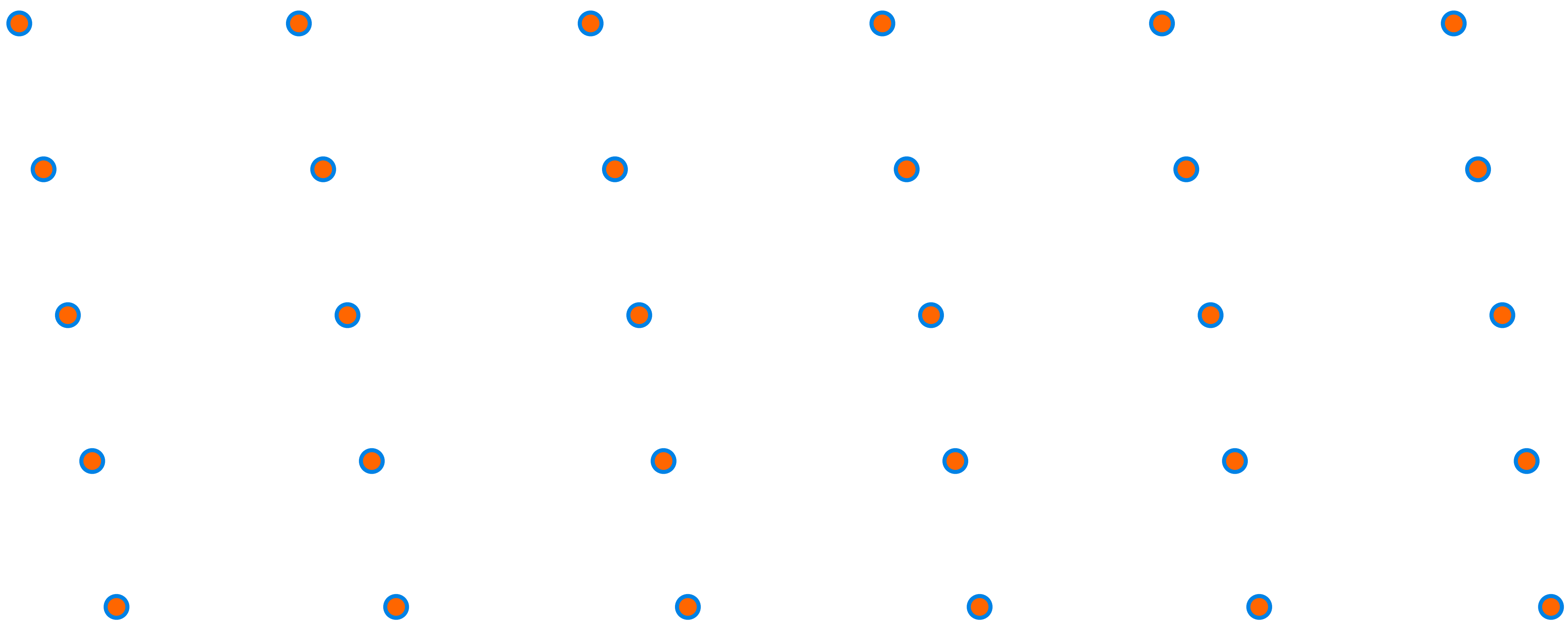
Generate (x, y) in two equivalent ways



Same Distribution (Discrete Gaussian, Uniform) !

What do these trapdoors look like?

Lattice Trapdoors (Type 1): Geometric View



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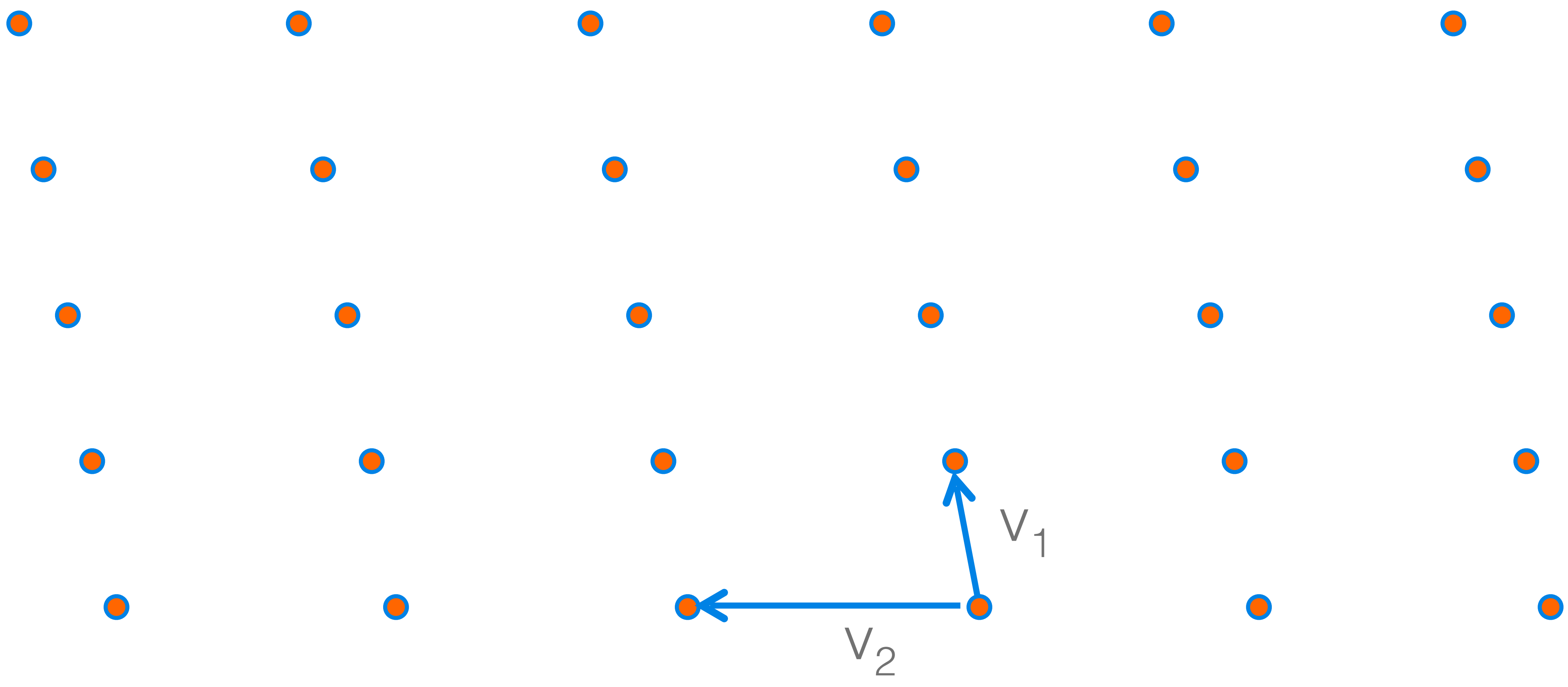

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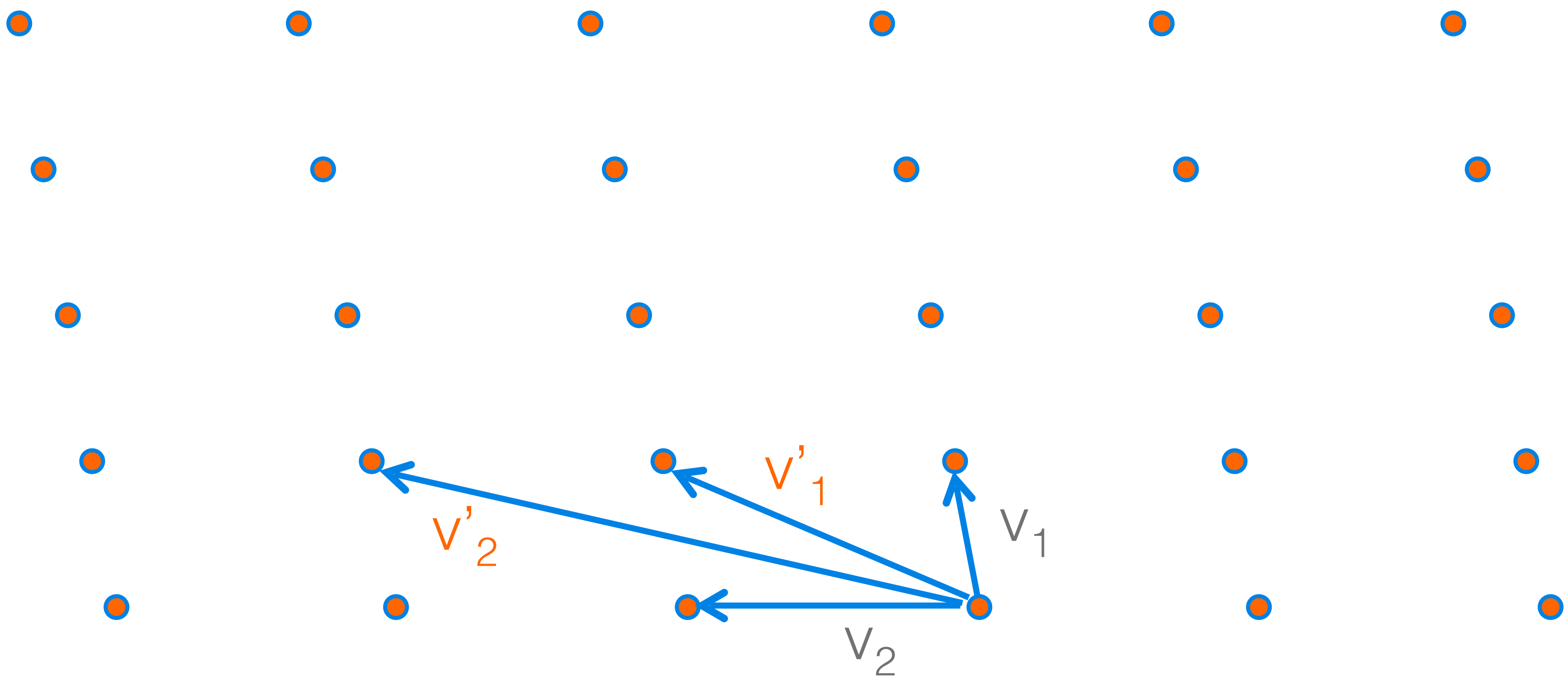

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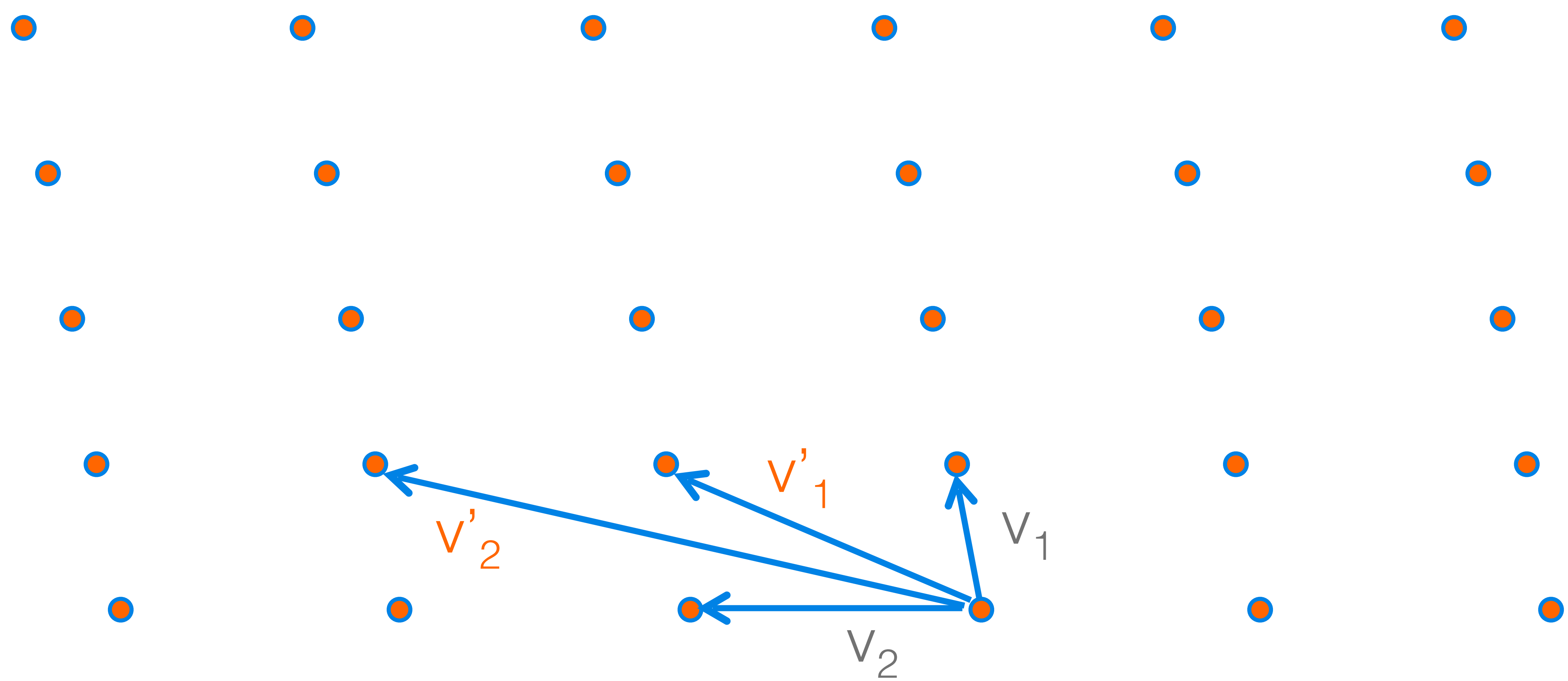
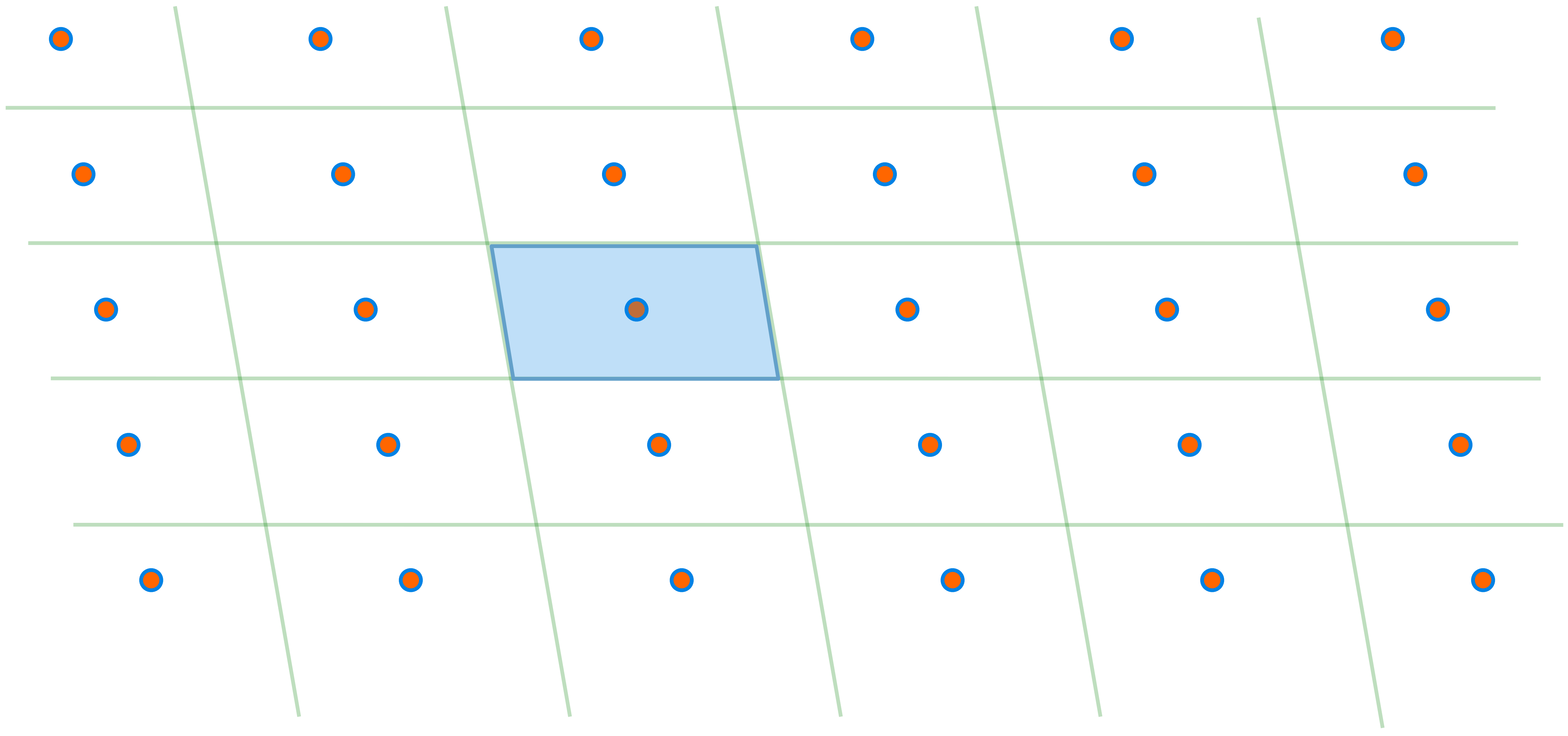


Multiple Bases

Parallelopipeds



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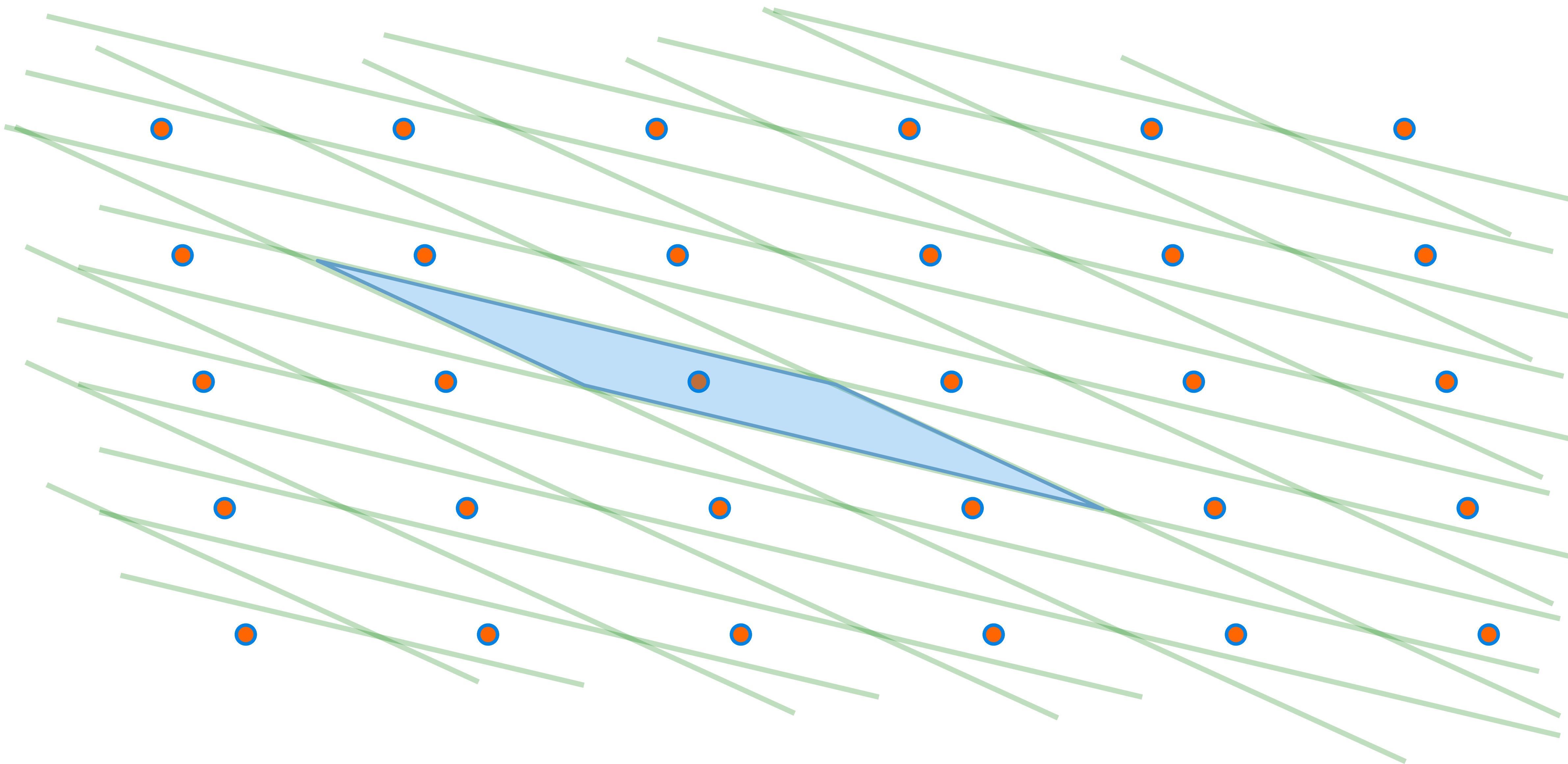
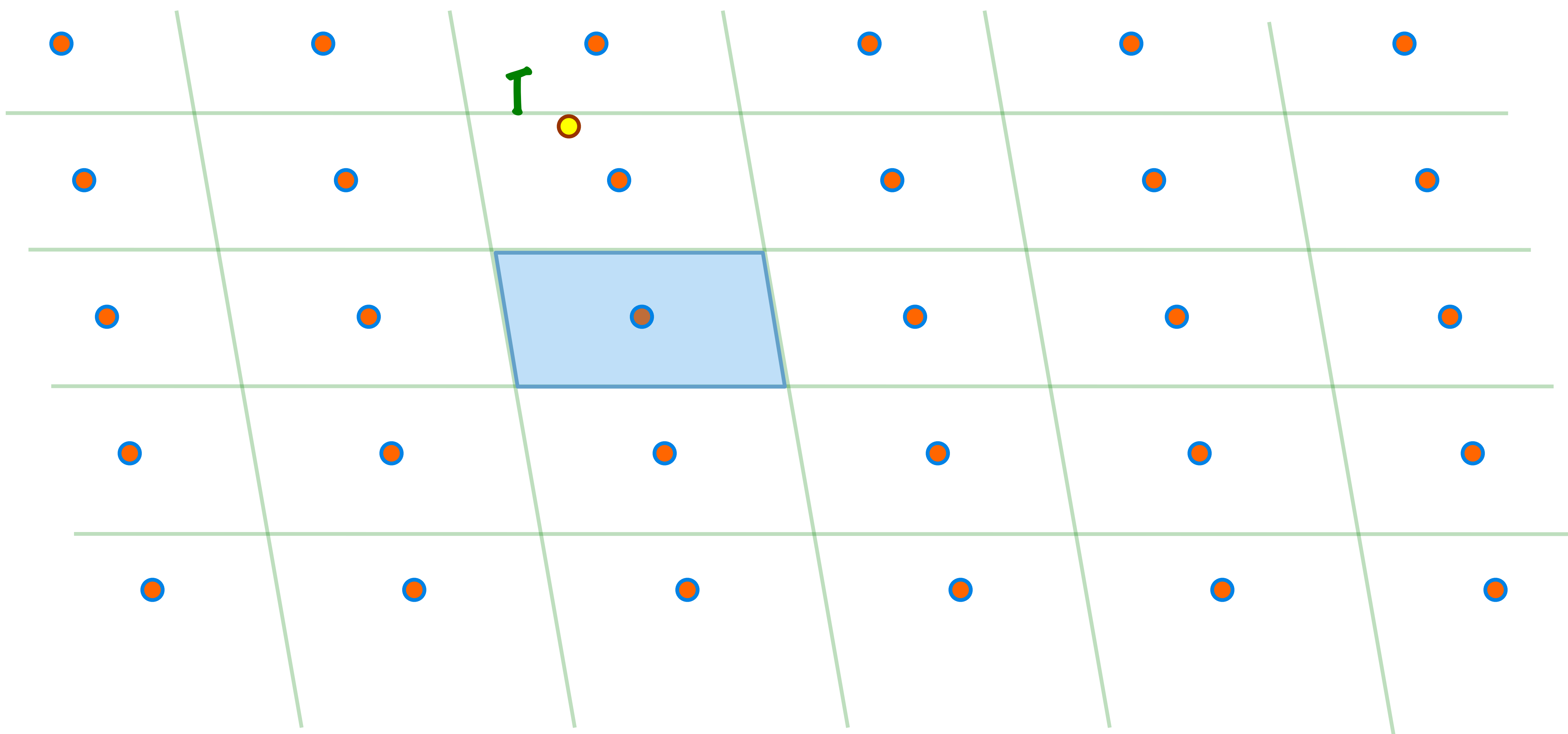


Good Basis



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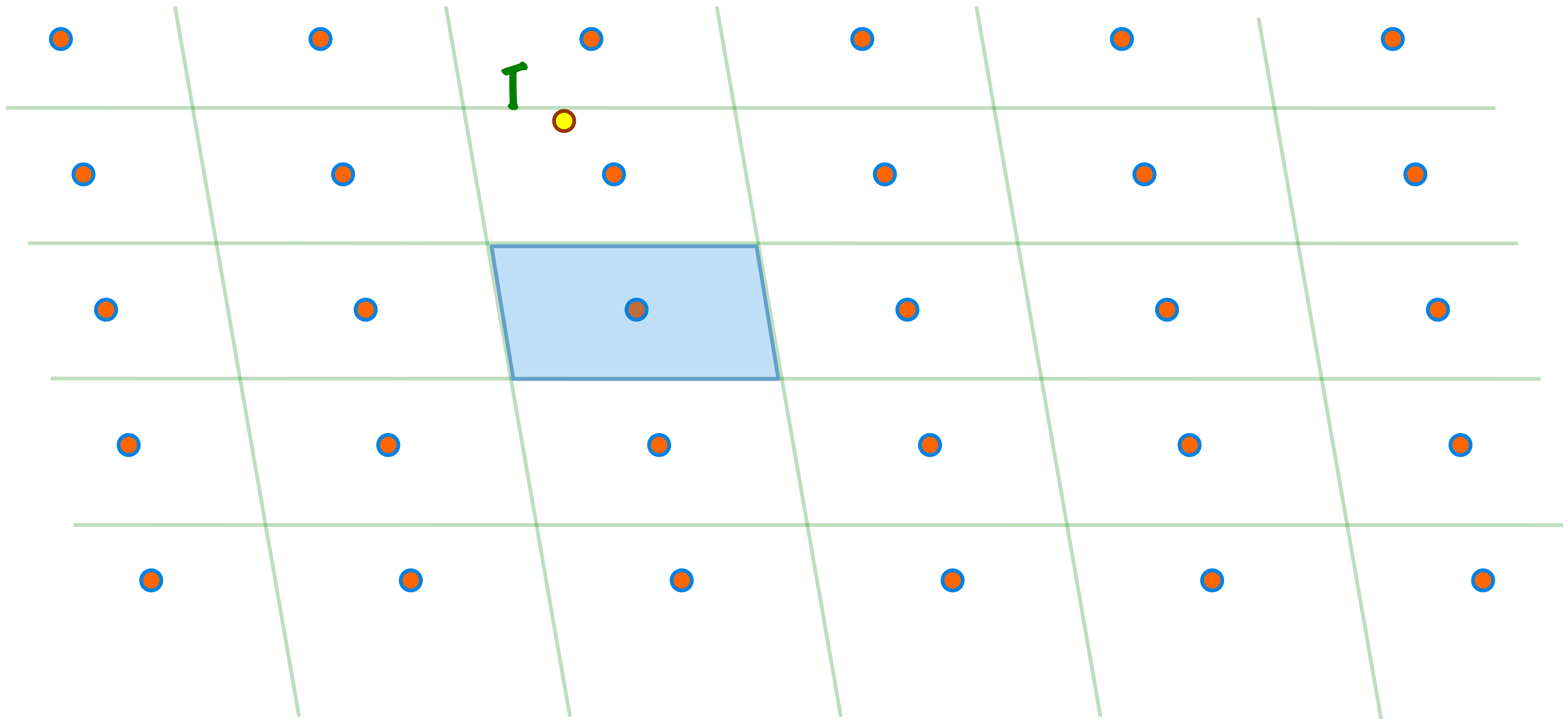

“Quite short” and “nearly orthogonal”

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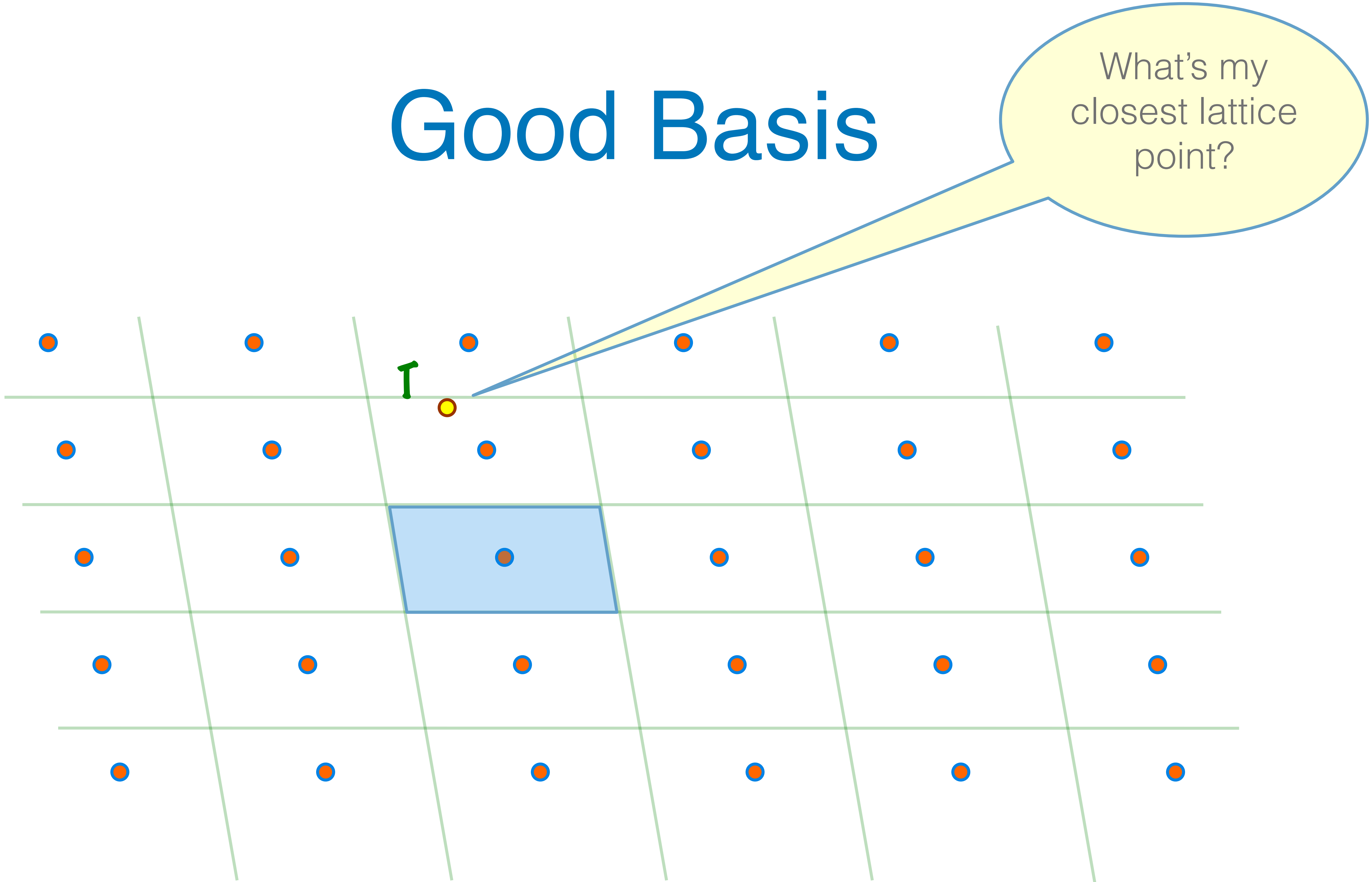

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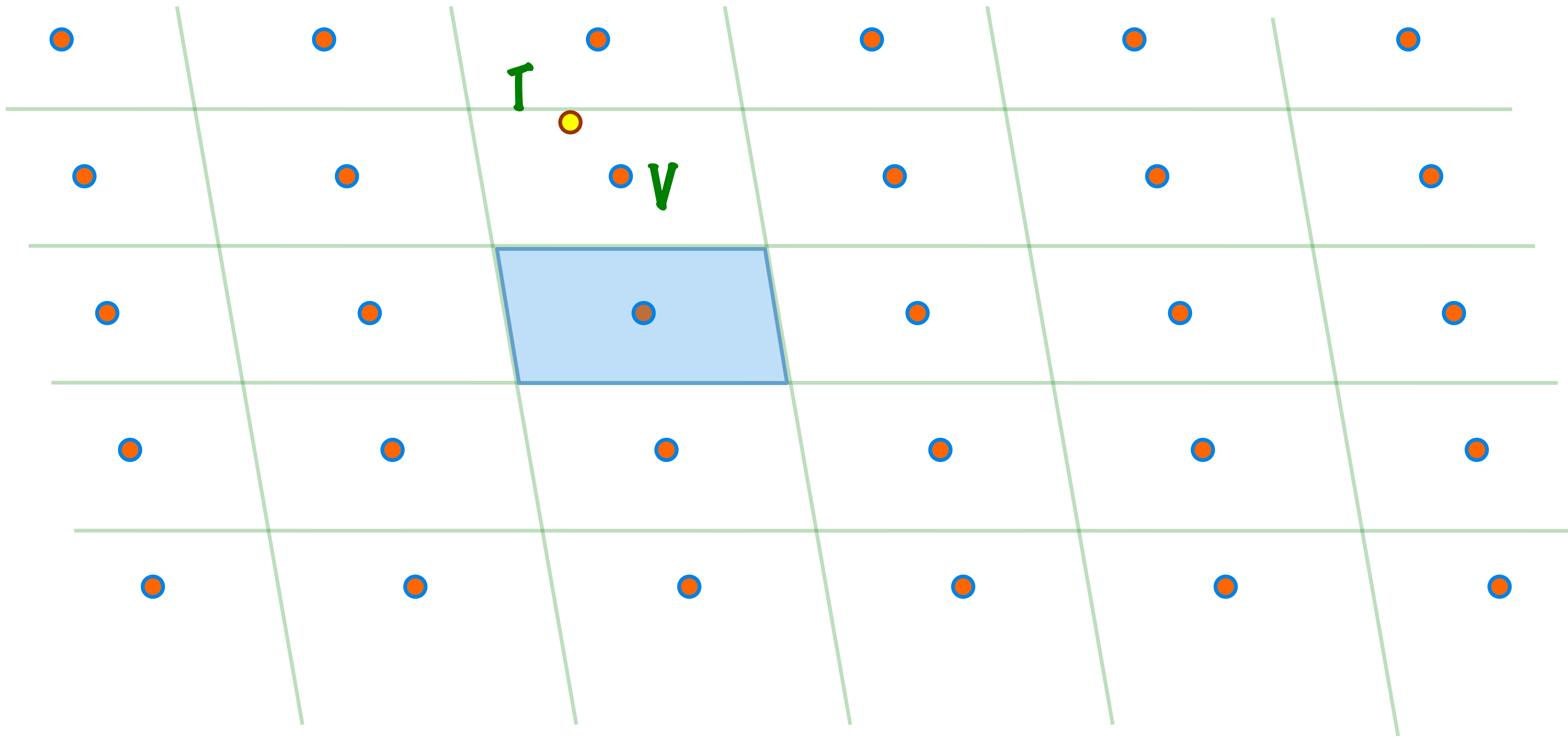

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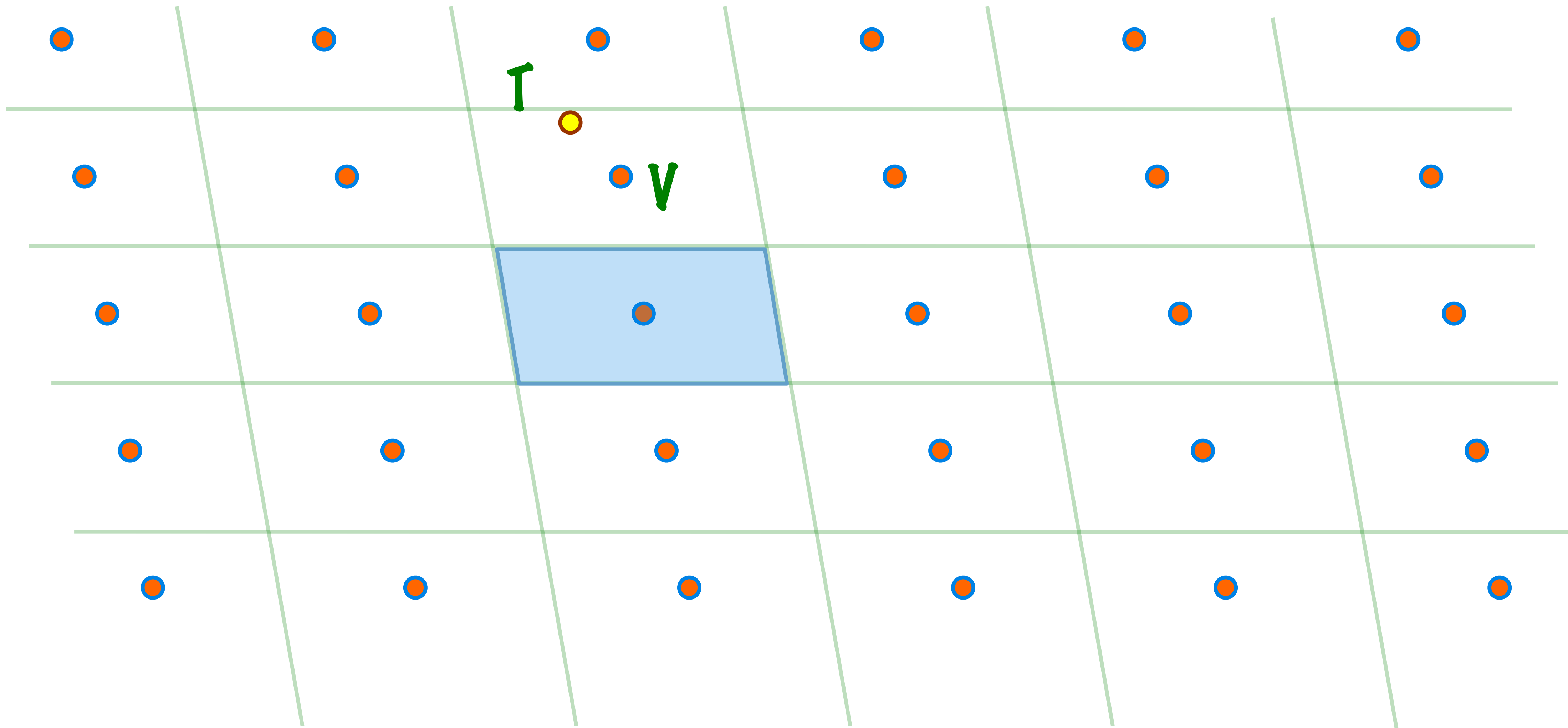

Output center of parallelopipid containing T

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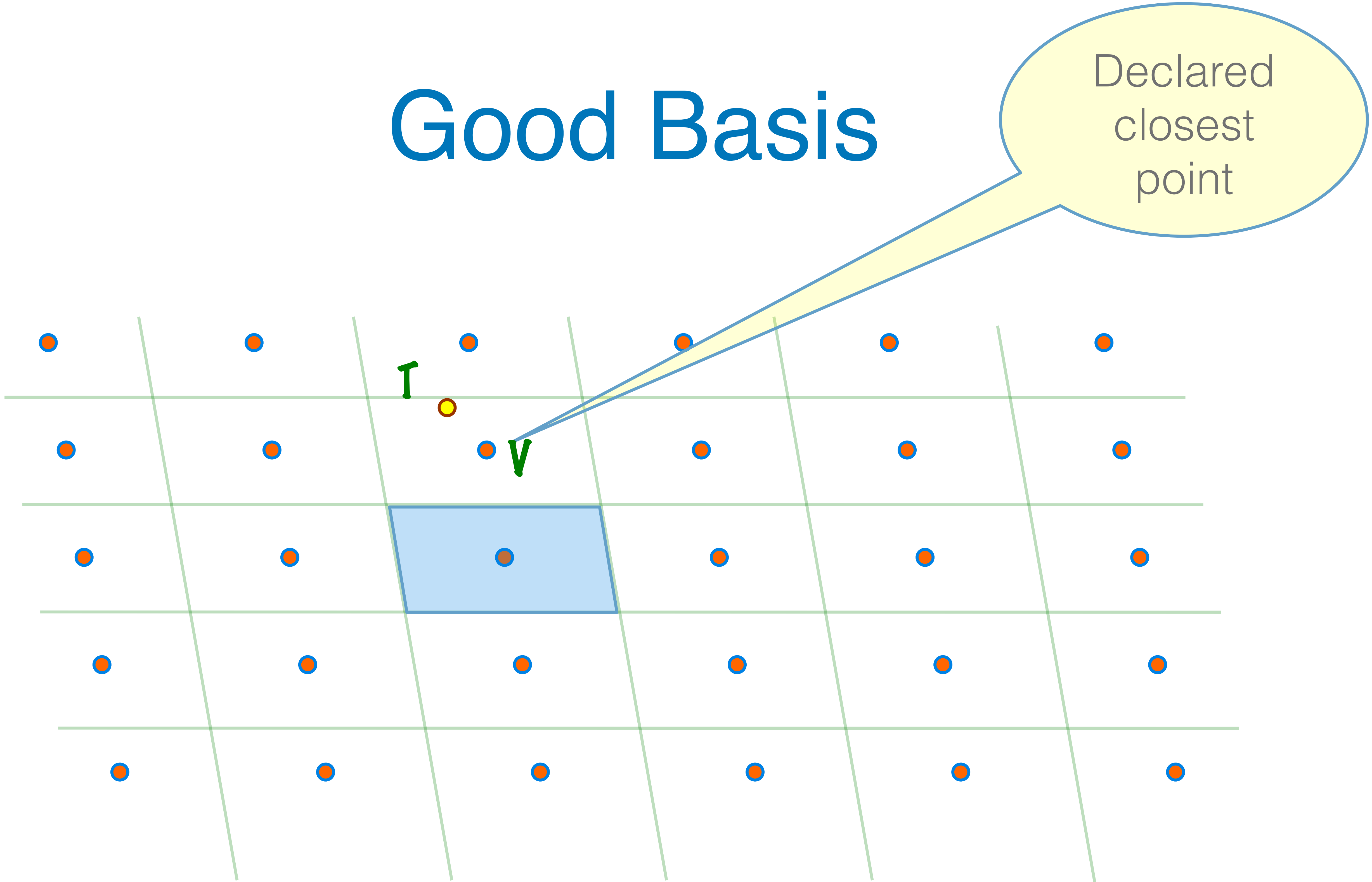

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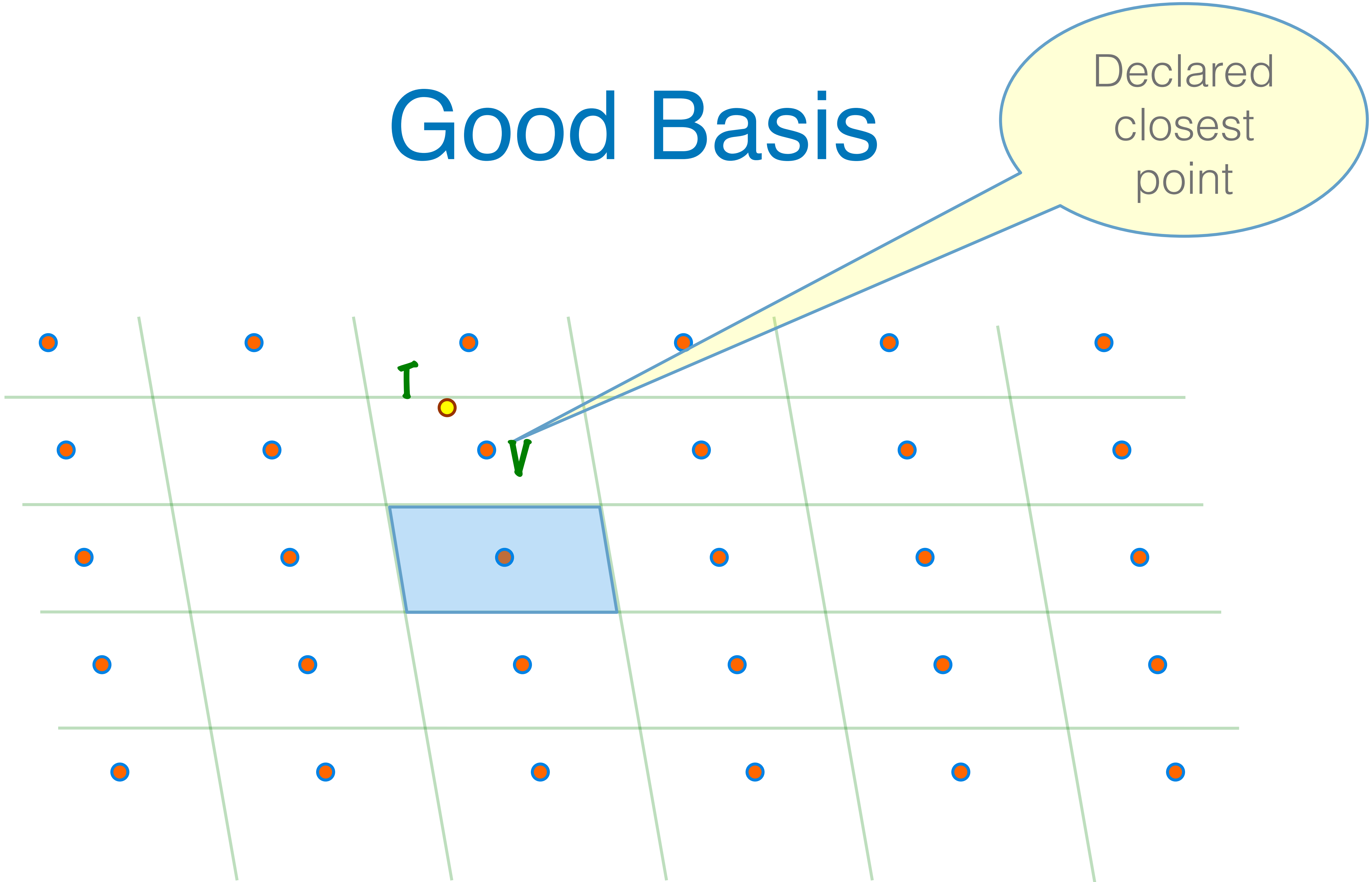

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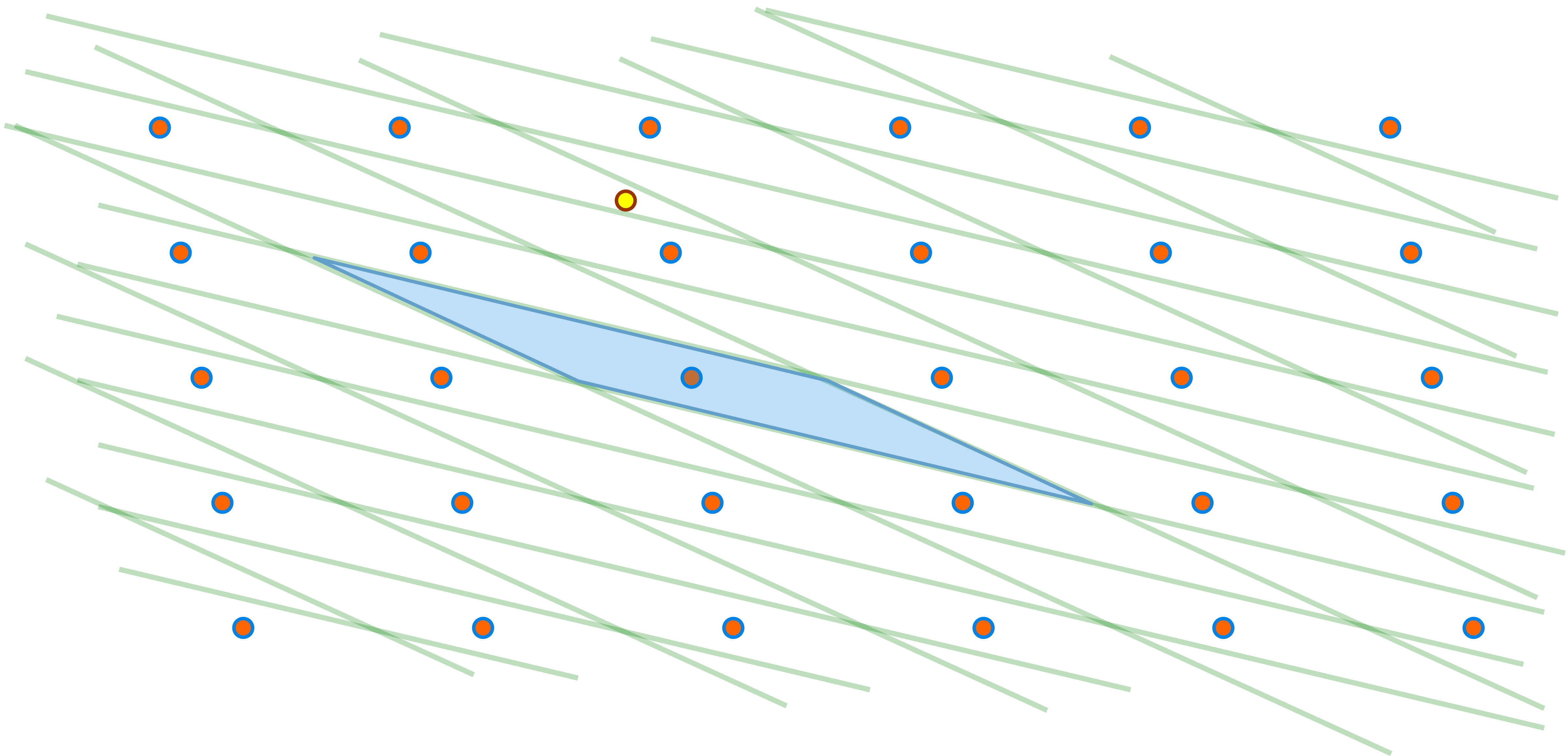
Pretty Accurate…

Bad Basis



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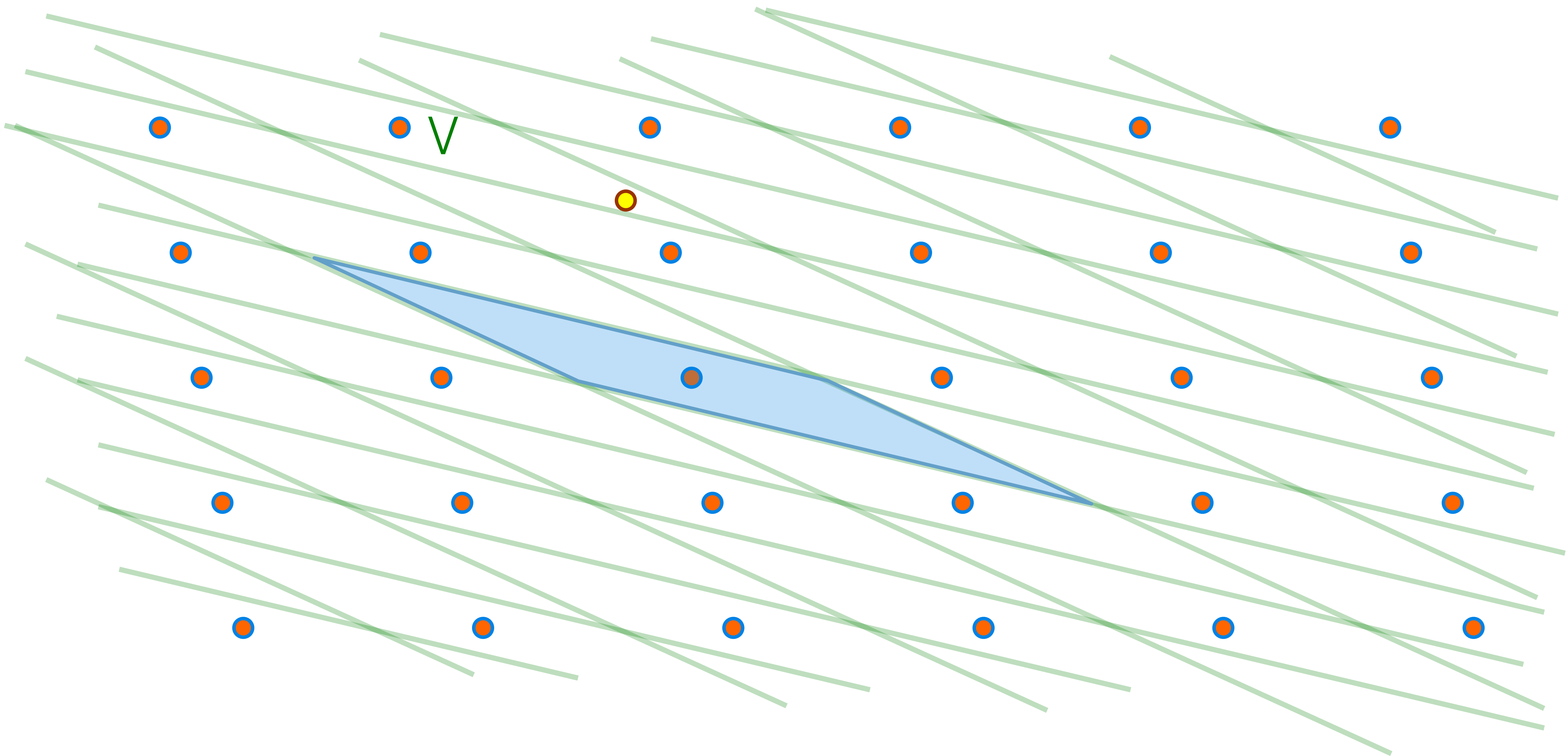

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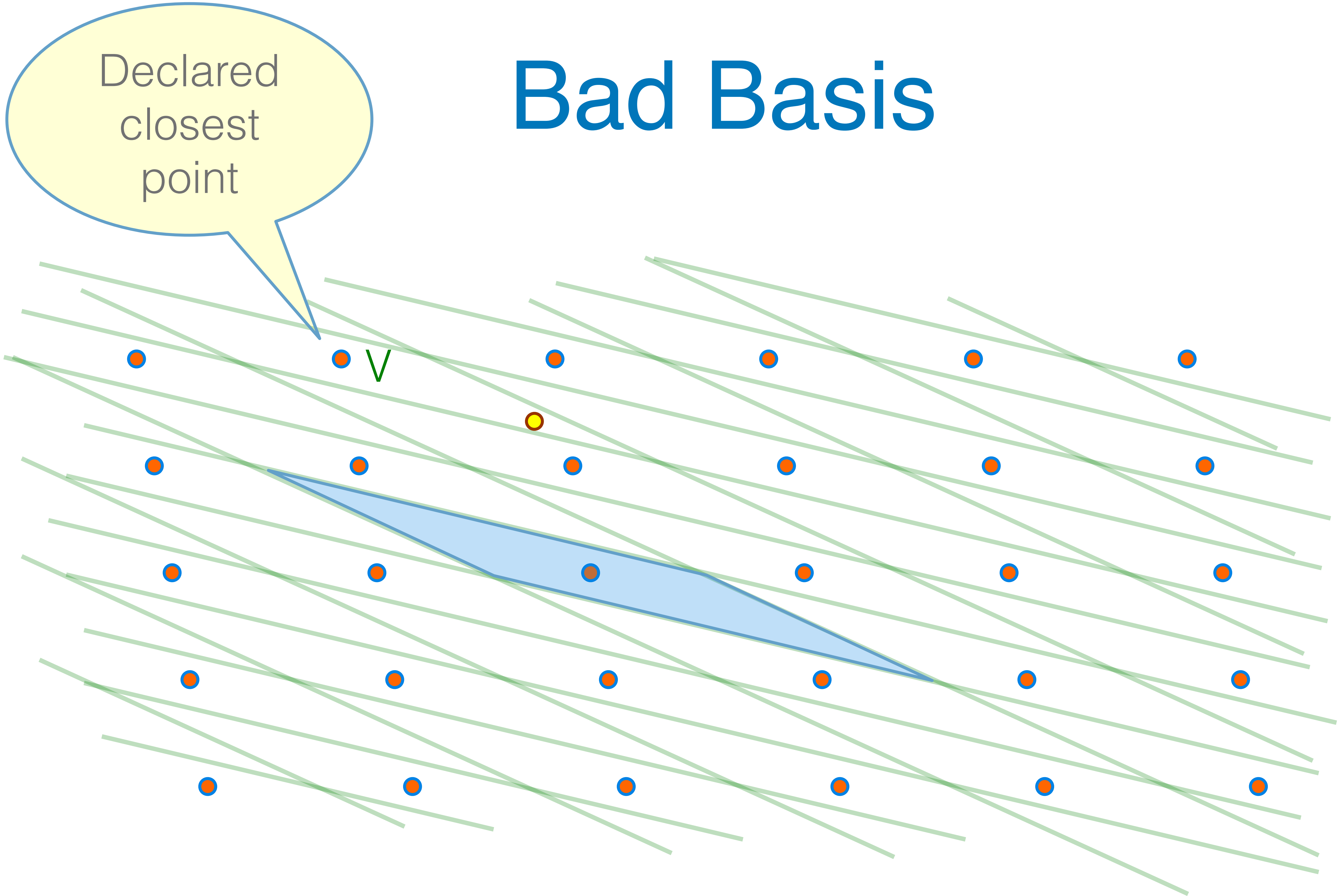

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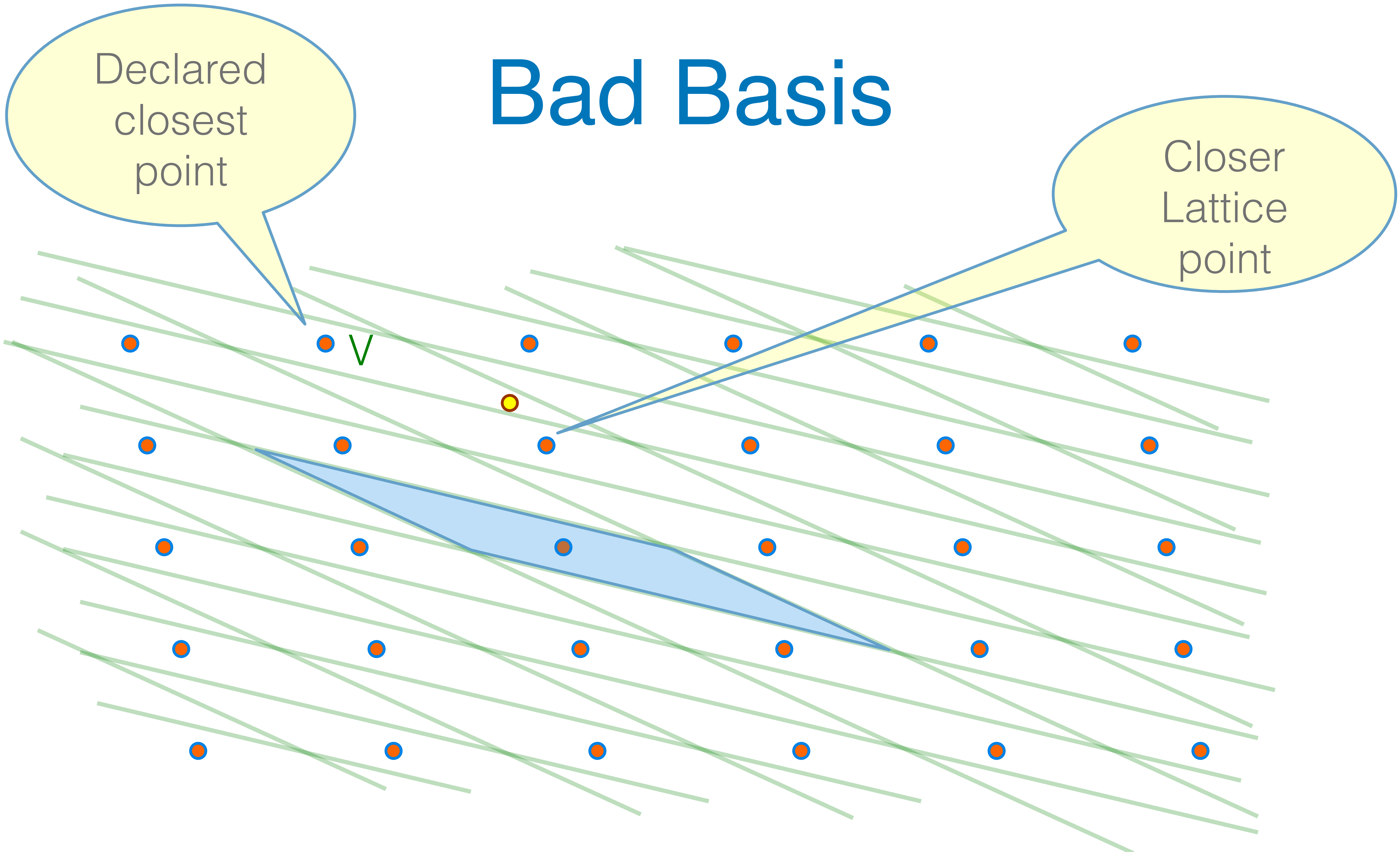

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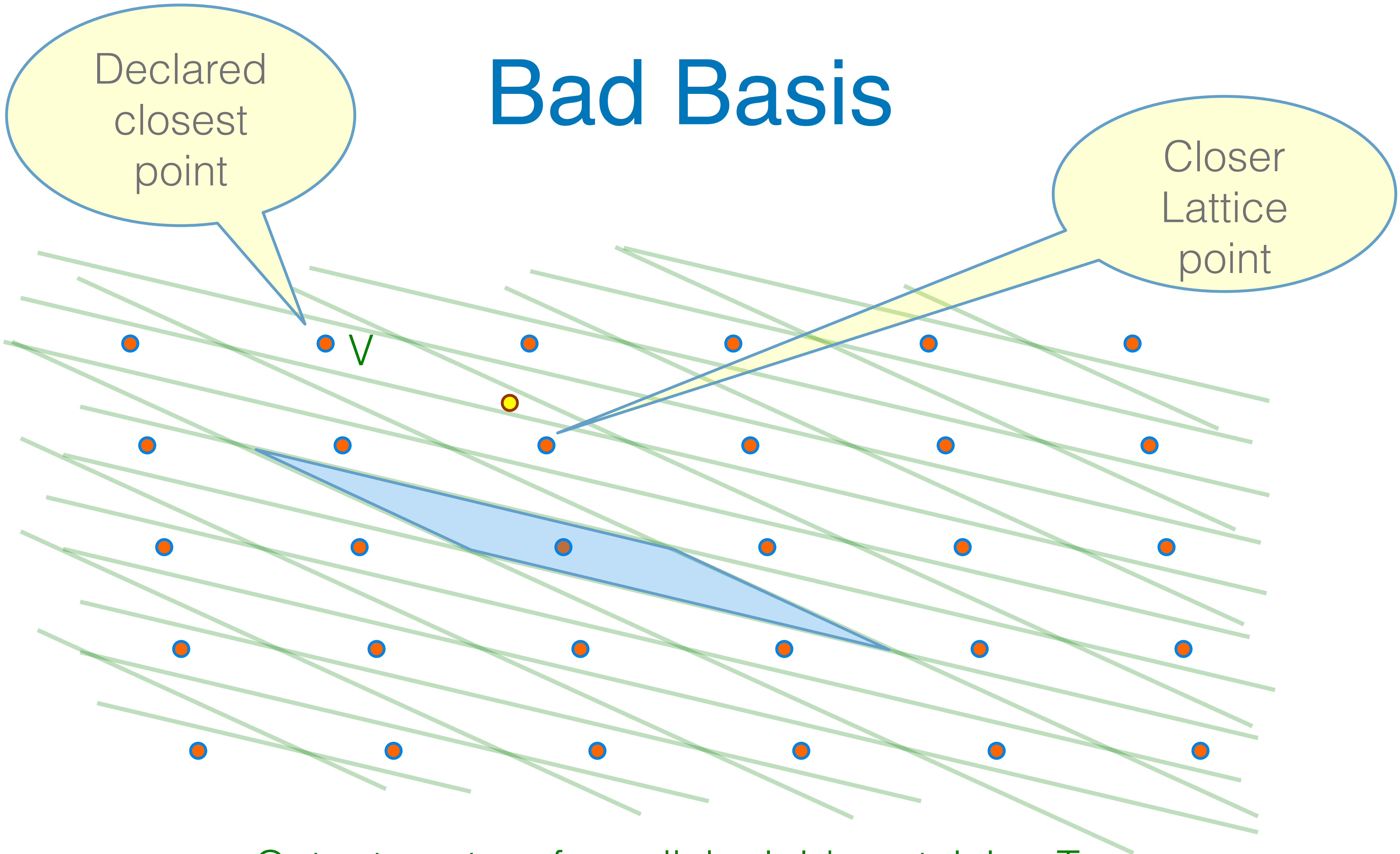

Output center of parallelopipid containing T

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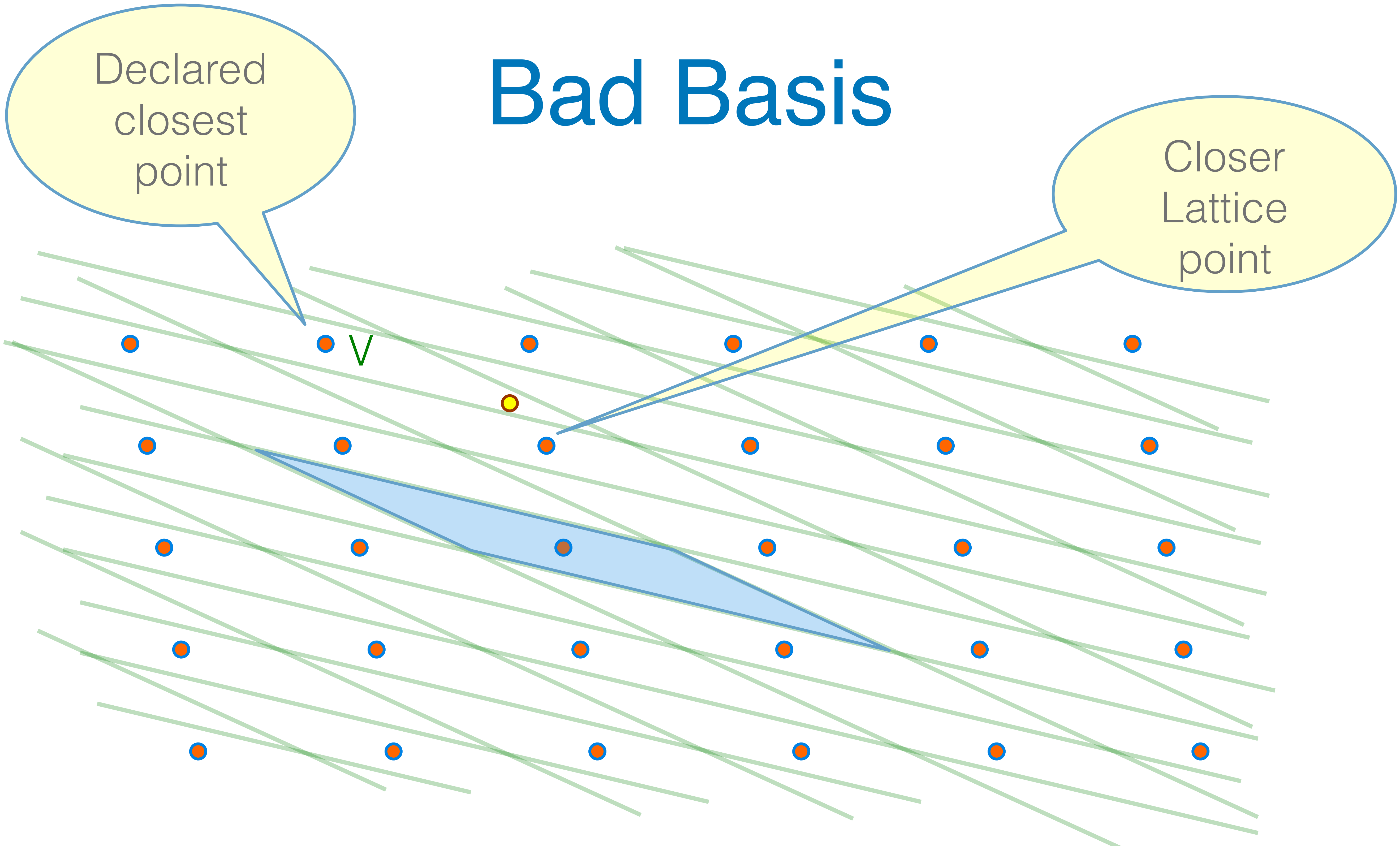

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Not So Accurate…

Basis quality and Hardness

- SVP, CVP, SIS (...) hard given arbitrary (bad) basis
- Some hard lattice problems are easy given a good basis
- Will exploit this asymmetry

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Use Short Basis as Cryptographic Trapdoor!

Lattice Trapdoors (Type 1)

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Inverting Our Function

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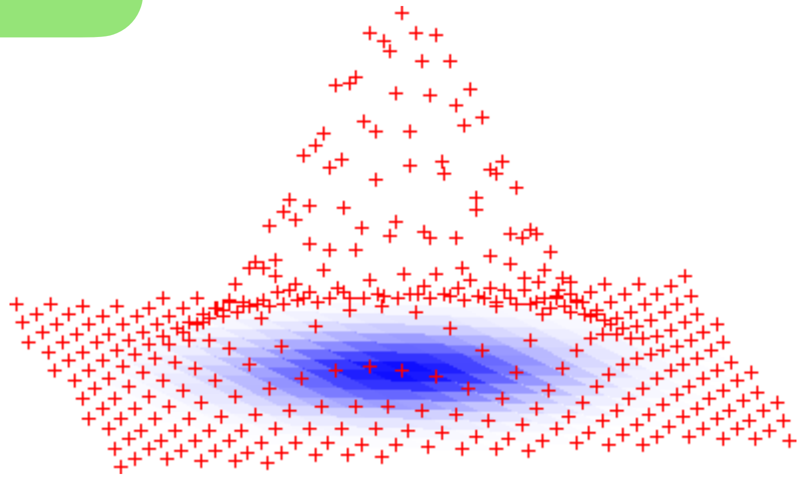

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$$\mathbf{x}' \leftarrow = f_{\mathbf{A}}^{-1}(\mathbf{u})$$

with prob $\propto \exp(-\|\mathbf{x}'\|^2/\sigma^2)$

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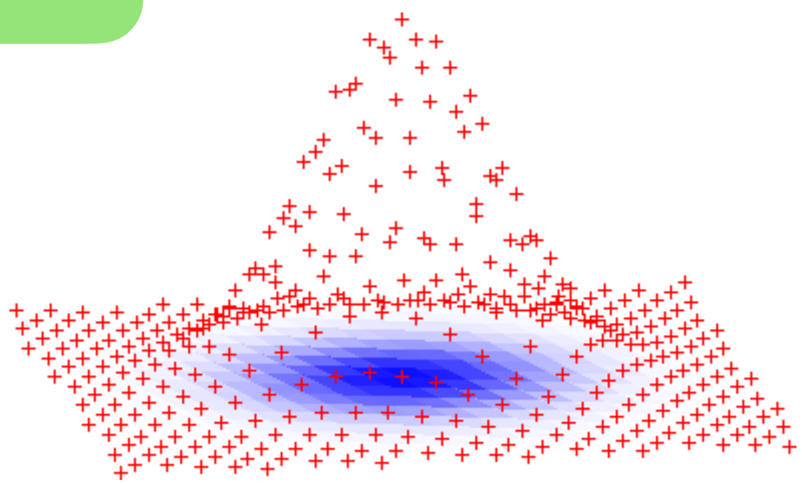

The Lattice

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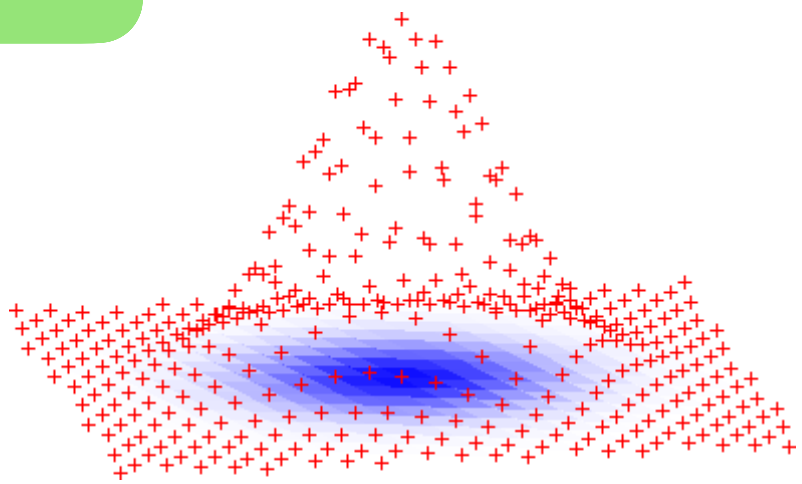

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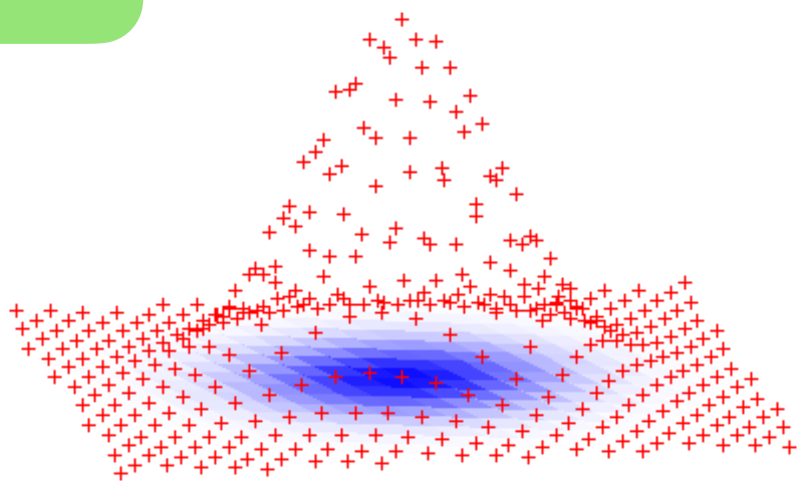

The Lattice

$$\boldsymbol{\Lambda} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = 0 \mod q\} \subseteq \mathbb{Z}_q^m$$

Short basis for $\boldsymbol{\Lambda}$ lets us sample from $f_{\mathbf{A}}^{-1}(\mathbf{u})$ with correct distribution!

Two Questions

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1. How to get short basis

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2. How to use short basis

Lattice Trapdoors (Type 2)

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- Implies Type 1 trapdoors

Type 2 Trapdoors [MP12]

Recall $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\ \mathbf{x} \mod q \in \mathbb{Z}_q^n$ and $g_{\mathbf{A}}(\mathbf{s}, \mathbf{e}) = \mathbf{s}^t \mathbf{A} + \mathbf{e}^t \mod q \in \mathbb{Z}_q^m$

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(fixed, public, offline)

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Randomize G ↔ A via nice unimodular transformation

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Transformation in Step 2 is the trapdoor!

Step 1: $f_{\mathbf{G}}^{-1},\ g_{\mathbf{G}}^{-1}$ for Gadget G

Recall $f_{\mathbf{G}}(\mathbf{x}) = \mathbf{G}\ \mathbf{x} \mod q \in \mathbb{Z}_q^n$ and $g_{\mathbf{G}}(\mathbf{s}, \mathbf{e}) = \mathbf{s}^t\mathbf{G} + \mathbf{e}^t \mod q \in \mathbb{Z}_q^m$

Let $q = 2^k$ and $\mathbf{g} = [1, 2, 4, \cdots, 2^{k-1}] \in \mathbb{Z}_q^{1\times k}$

Invert LWE: find $s \in Z_q$ **s.t.** $s \cdot \mathbf{g} + \mathbf{e} = [s + e_0, 2s + e_1, \ \cdots\ 2^{k-1}s + e_{k-1}]$

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- Then get next bit of s and so on.
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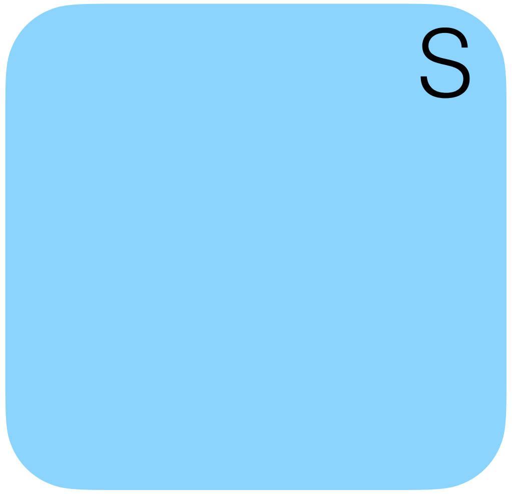
Gaussian from shifted lattice 2Z + u

Invert SIS: sample Gaussian preimage **x** s.t. $u = \langle \mathbf{g}\ \mathbf{x} \rangle \mod q$

- For $i \in [0, \ldots, k-1]$, choose $x_i \leftarrow (2\mathbb{Z} + u)$, $u \leftarrow (u - x_i)/2 \in \mathbb{Z}$
- Let k= 2. $x_0 \leftarrow (2z_0 + u)$, $u \leftarrow (u - 2z_0 - u)/2 = -z_0$

$$x_1 \leftarrow (2z_1 - z_0)$$
$$\langle \mathbf{g}, \mathbf{x} \rangle = 2z_0 + u + 2(2z_1 - z_0) = u + 4z_1 = u \mod 4$$

Step 1: $f_{\mathbf{G}}^{-1}$, $g_{\mathbf{G}}^{-1}$ for Gadget G

Want $\mathbf{g} = [1, 2, 4, \cdots, 2^{k-1}]$ 

$=$ 0 mod q

Step 1: $f_{\mathbf{G}}^{-1},\ g_{\mathbf{G}}^{-1}$ for Gadget G

Note $\mathbf{g} = [1, 2, 4, \cdots, 2^{k-1}]$

$$\underbrace{\begin{bmatrix} 2 & & & & \\ -1 & 2 & & & \\ & -1 & 2 & & \\ & & \ddots & \ddots & \\ & & & 2 & \\ & & & -1 & 2 \end{bmatrix}}_{S} = 0 \bmod q$$

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$f_{\mathbf{G}}^{-1},\ g_{\mathbf{G}}^{-1}$ reduce to n parallel, offline calls to $f_{\mathbf{g}}^{-1},\ g_{\mathbf{g}}^{-1}$

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A is uniform by leftover hash lemma!

Leftover Hash Lemma (oversimplified)

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Hence **A** = $\left[\, \mathrm{B} \;\middle|\; \mathrm{G} - \mathrm{BR} \,\right]$ uniform

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Basis S **for** $\Lambda^{\perp}(\mathbf{G})$

&

Trapdoor R for A

→

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Basis $\mathbf{S}$ **for** $\Lambda^{\perp}(\mathbf{G})$ & **Trapdoor R for A** $\Rightarrow$ **Basis** $\mathbf{S_A}$ **for** $\Lambda^{\perp}(\mathbf{A})$

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Works if $\mathbf{e}^t \cdot \begin{bmatrix} \mathbf{R} \\ \mathbf{I} \end{bmatrix} \in [-q/4, q/4)$

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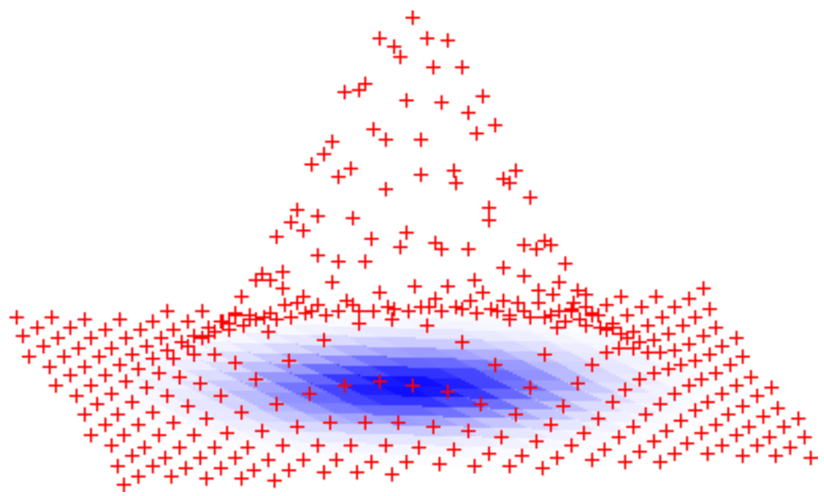

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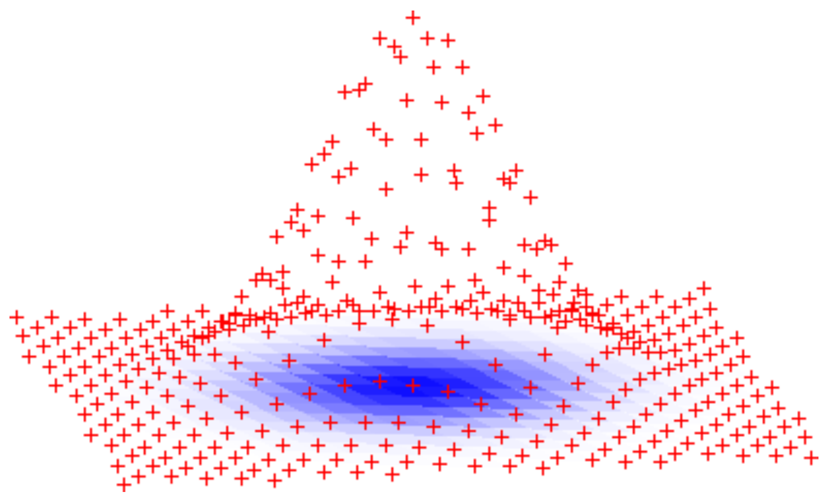

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Output $\mathbf{x} = \begin{bmatrix} \mathbf{R} \\ \mathbf{I} \end{bmatrix} \cdot \mathbf{z}$

Then,

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Covariance of x leaks R!

$$\Sigma := \mathbb{E}_{\mathbf{x}}\left[\mathbf{x} \cdot \mathbf{x}^t\right] = \mathbb{E}_{\mathbf{z}}\left[\mathbf{R} \cdot \mathbf{z}\mathbf{z}^t \cdot \mathbf{R}^t\right] \approx s^2 \cdot \mathbf{R}\mathbf{R}^t.$$

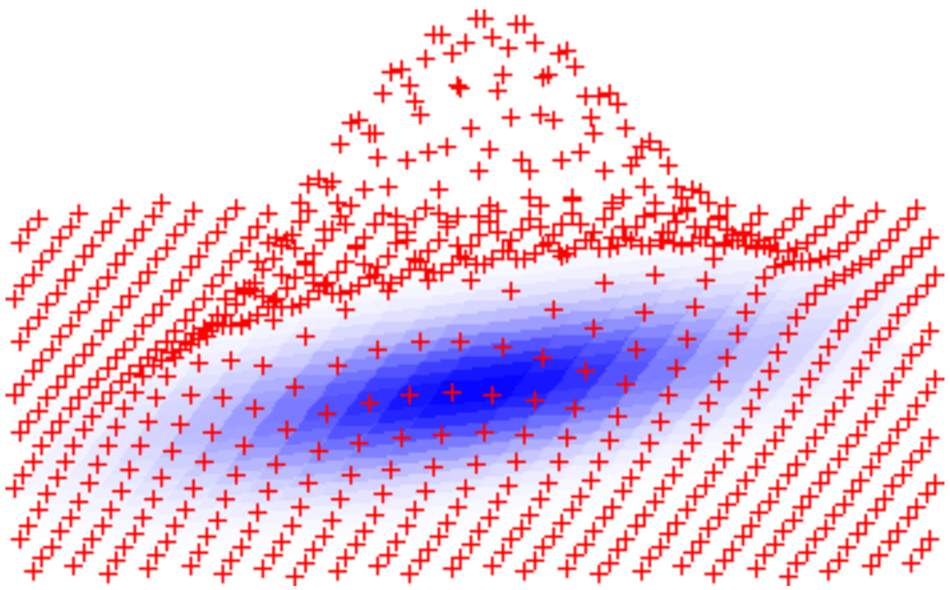

Image Credit: Chris Peikert

Step 3: Reduce $f_{\mathbf{A}}^{-1}$, $g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}$, $g_{\mathbf{G}}^{-1}$

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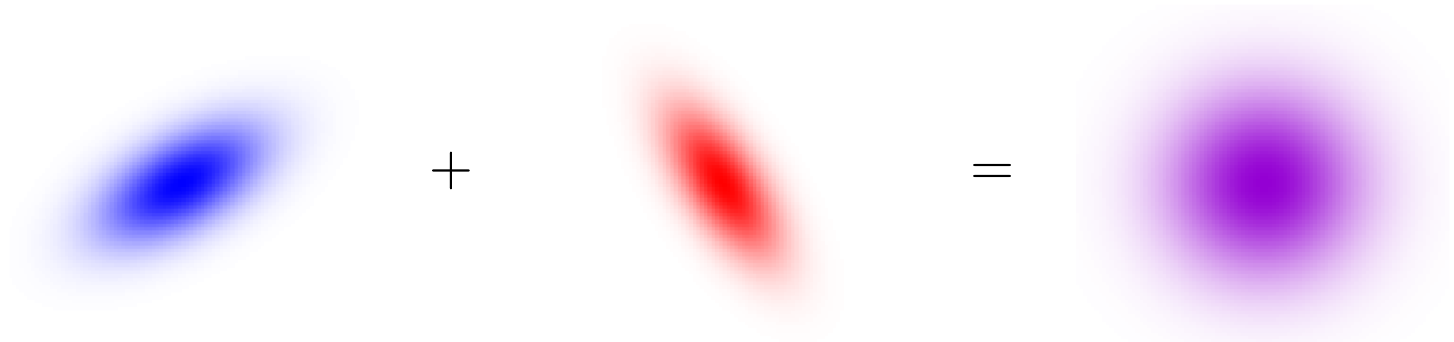

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Check

$$\mathbf{A} \cdot \begin{bmatrix} \mathbf{R} \\ \mathbf{I} \end{bmatrix} = \mathbf{G}$$

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{A}\mathbf{p} + \mathbf{A}\begin{bmatrix} \mathbf{R} \\ \mathbf{I} \end{bmatrix} \cdot \mathbf{z}$$

$$= \mathbf{A}\mathbf{p} + \mathbf{G}\mathbf{z} = \mathbf{u}$$

Takeaway for Applications

Let $\mathbf{B} \in \mathbb{Z}_q^{n \times m'}$, uniform $\mathbf{R} \in \mathbb{Z}_q^{m' \times n \log q}$, Gaussian

Let $\mathbf{A} = \left[\begin{array}{c|c} \text{B} & \text{G - BR} \end{array}\right]$

Then, $\mathbf{A}$ uniform, admits LWE and SIS inversion

$$f_{\mathbf{A}}^{-1}, \ g_{\mathbf{A}}^{-1}$$

Applications

Applications

A word about notation

Identity Based Encryption (IBE)

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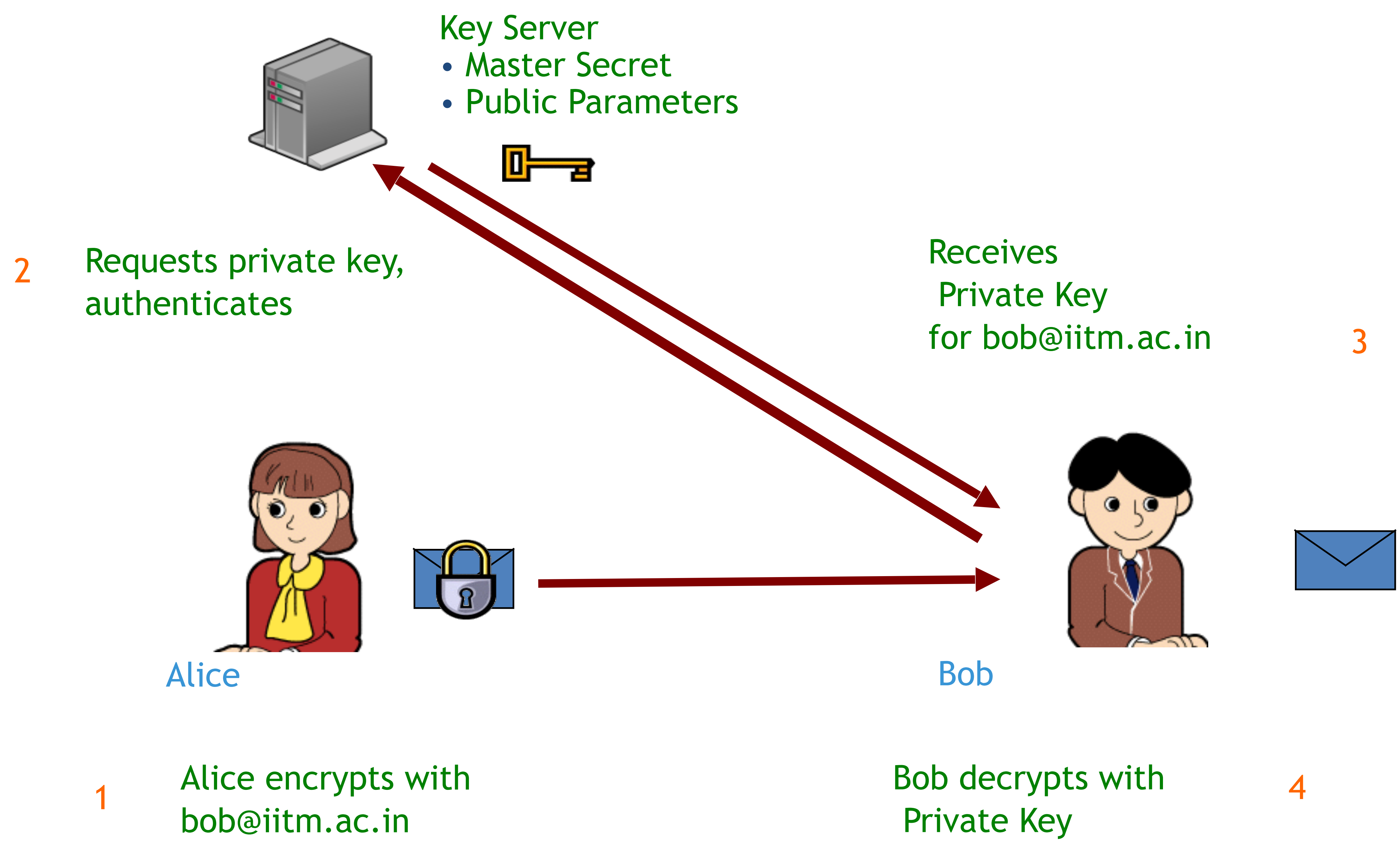
In short…………..

Identity Based Encryption (IBE)

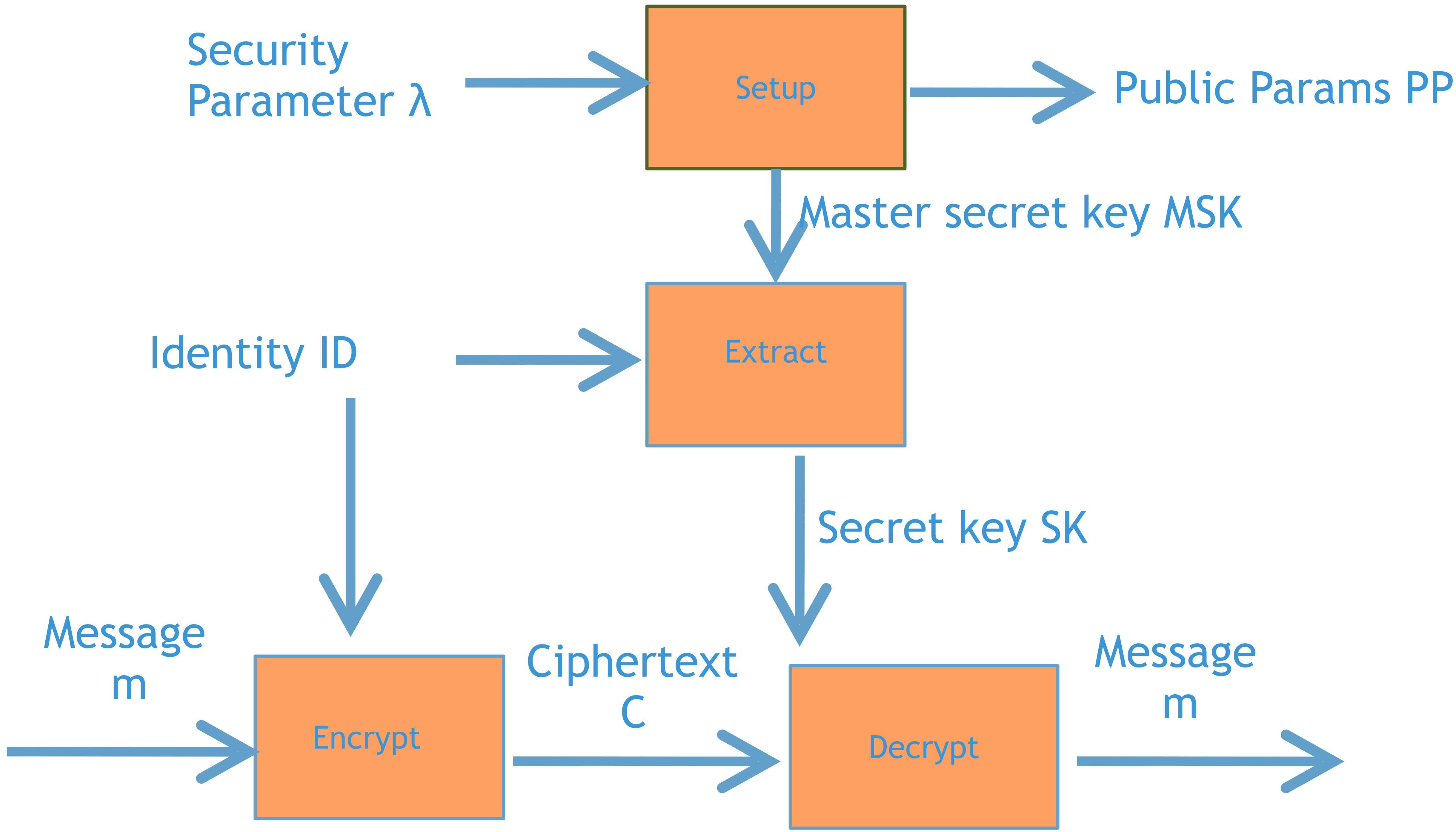
In short…………

Public Key Encryption in which ANY arbitrary string can be public key!

IBE: How does it work?



Identity Based Encryption



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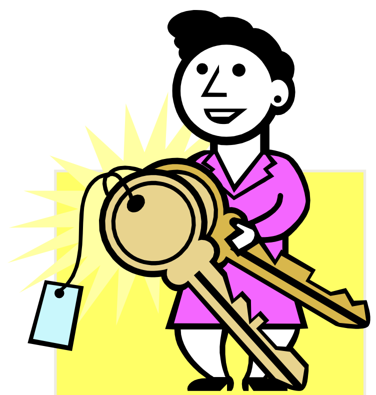
IBE Security



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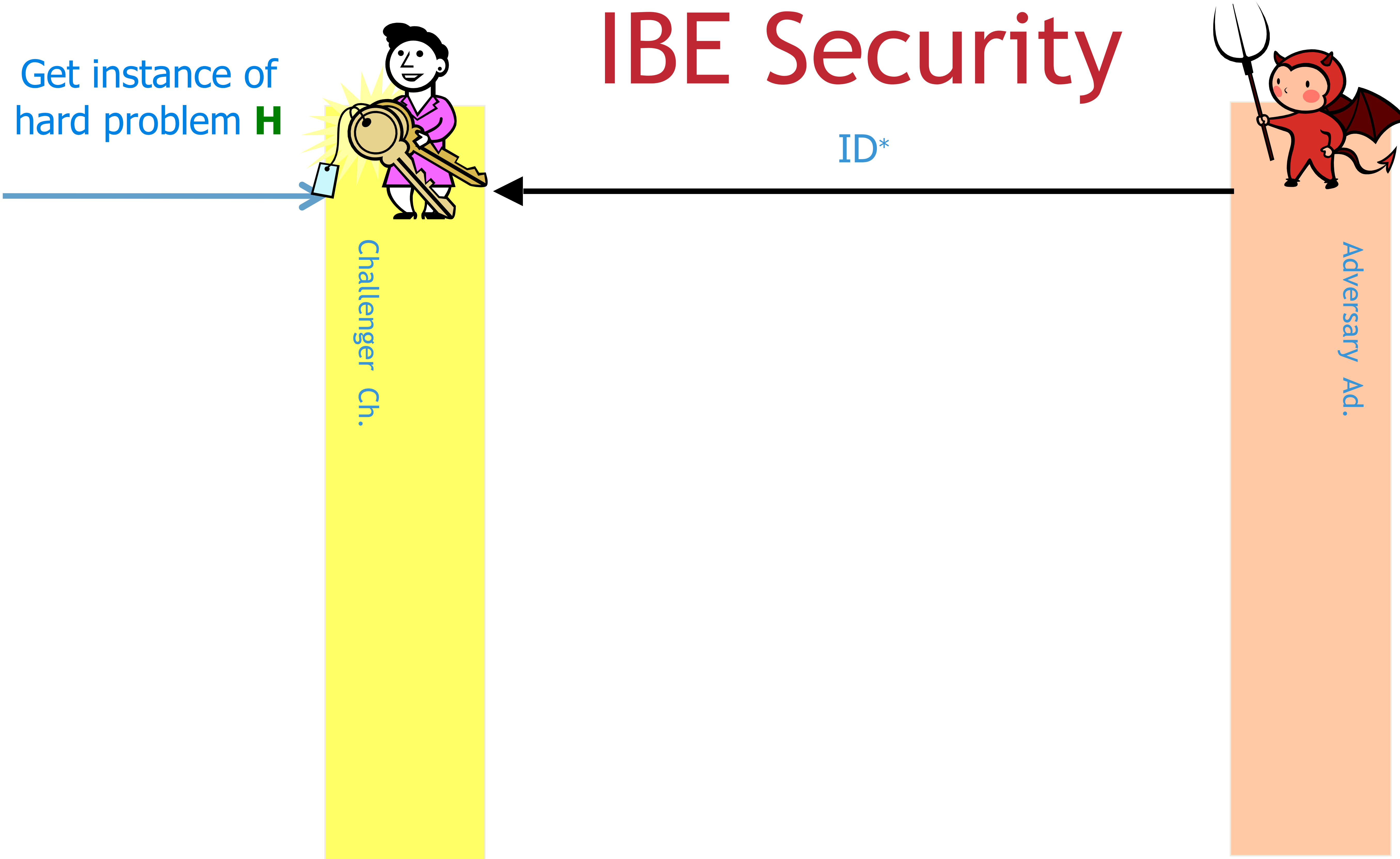

IBE Security

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ID*

PK

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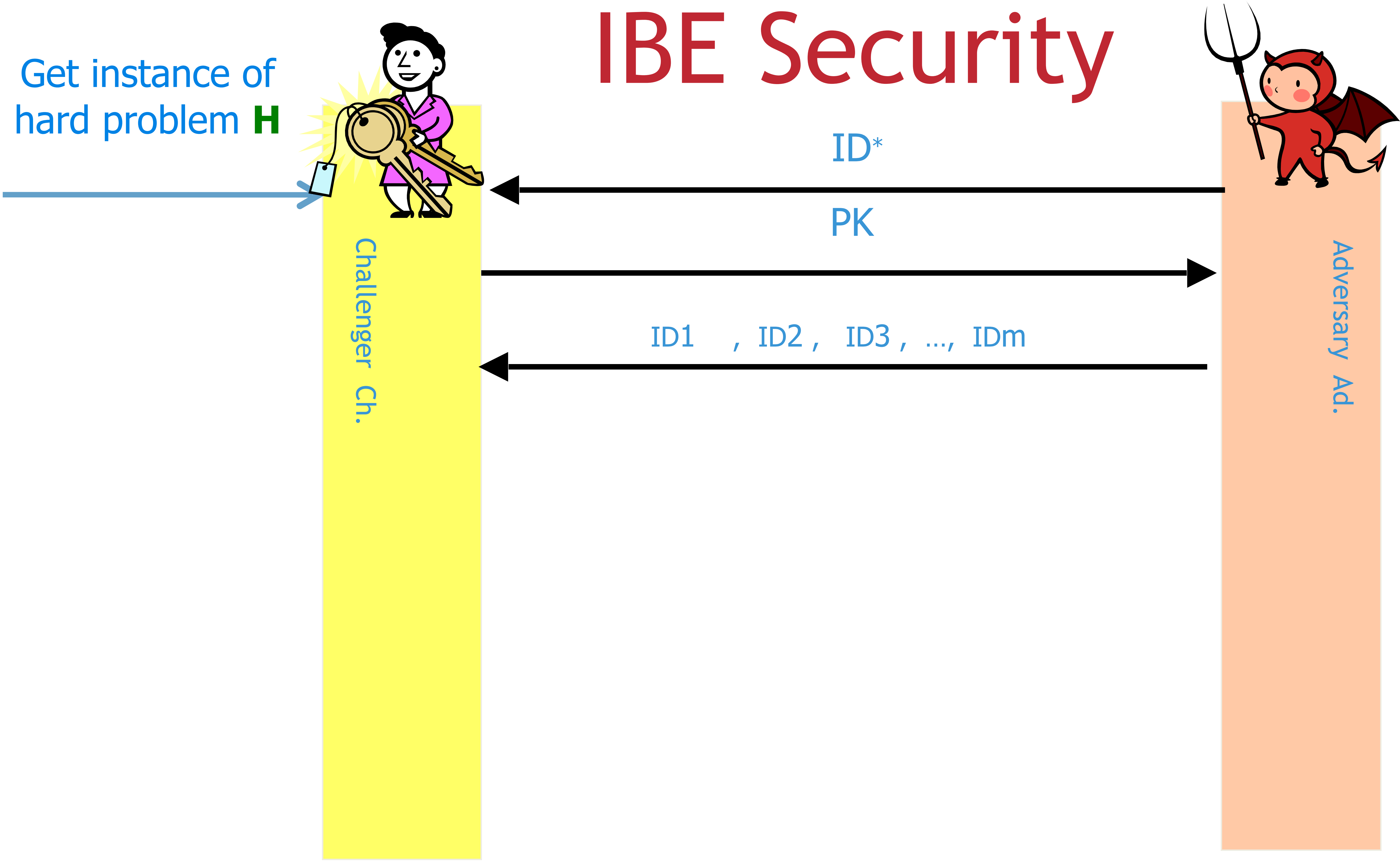

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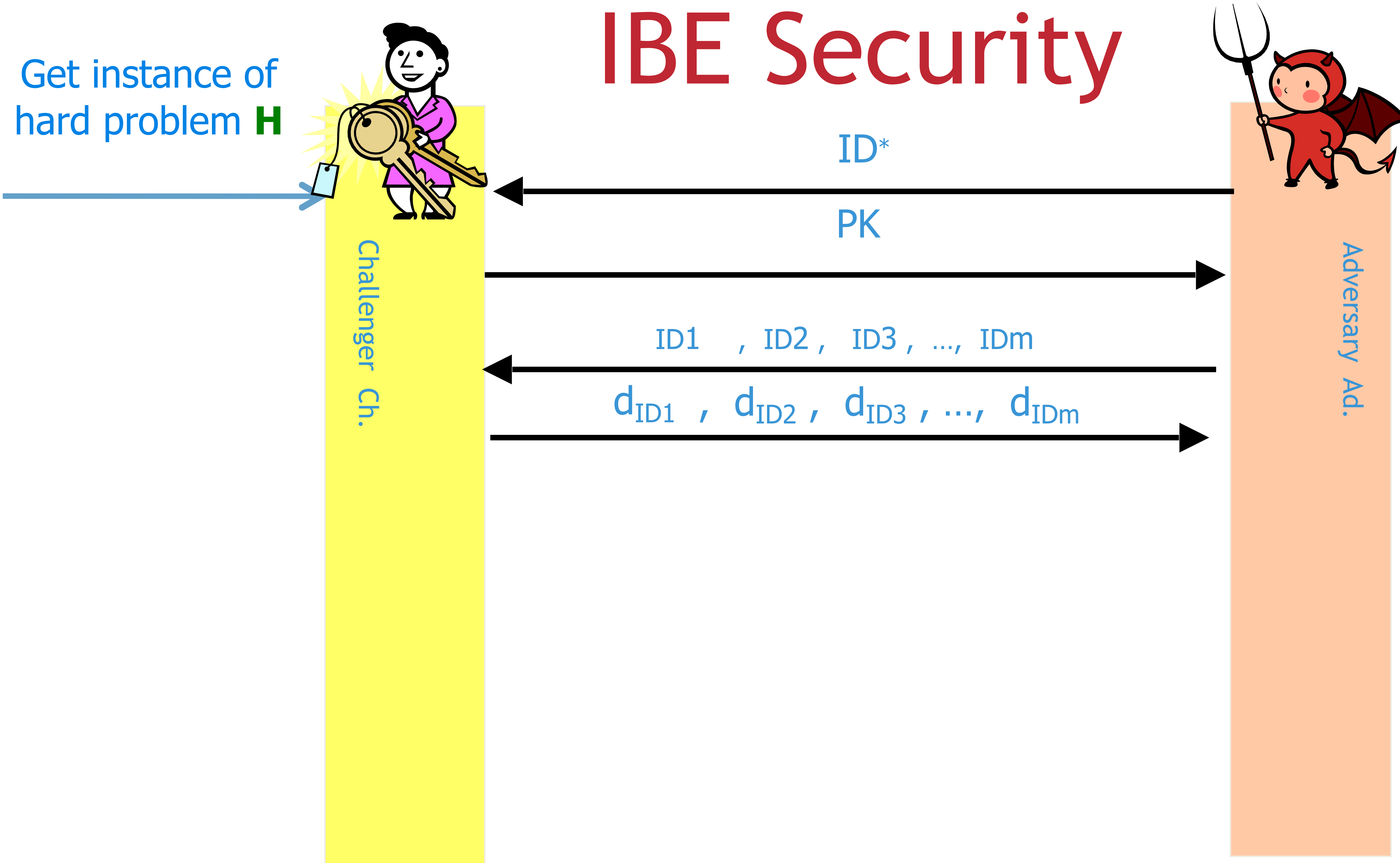

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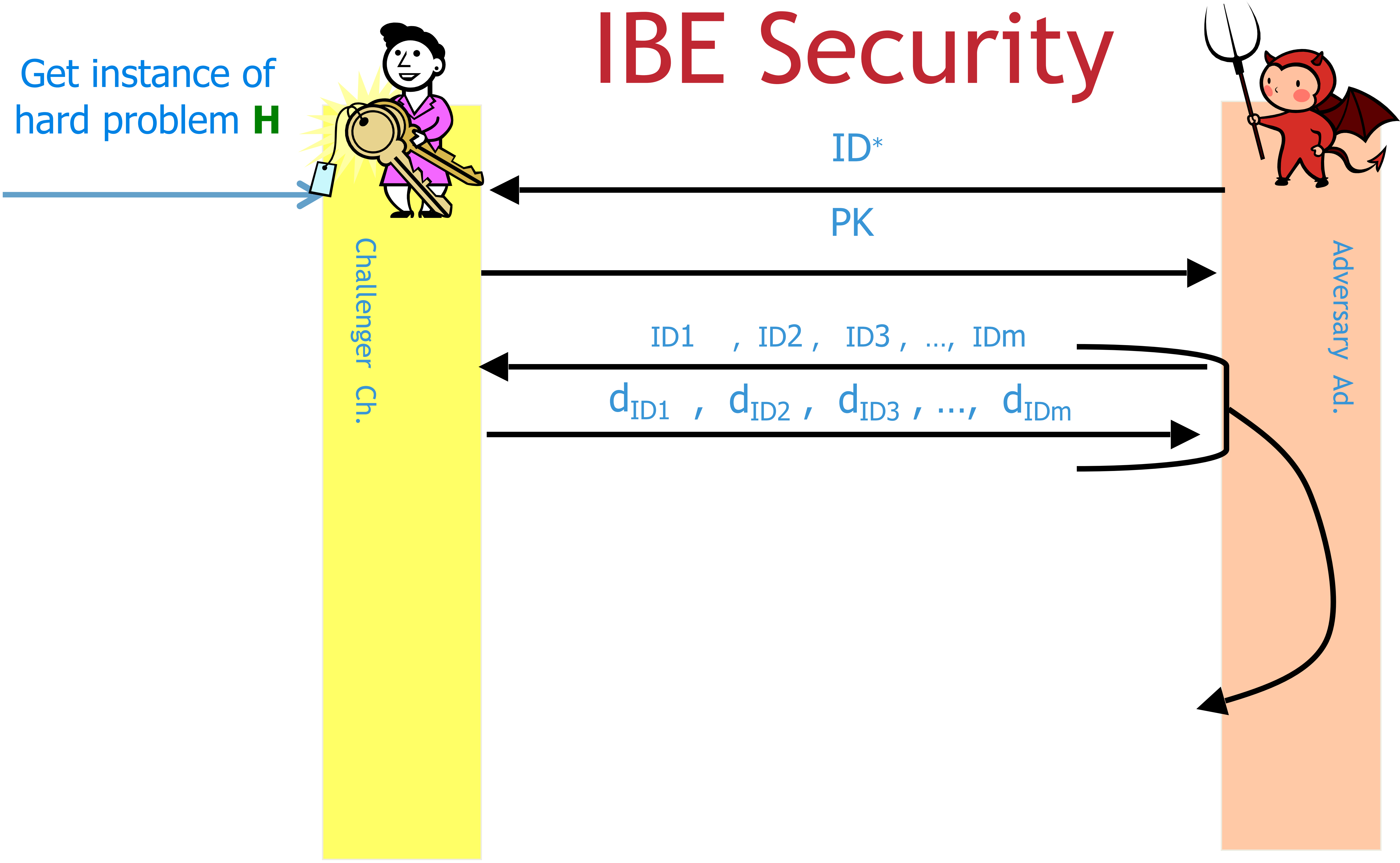

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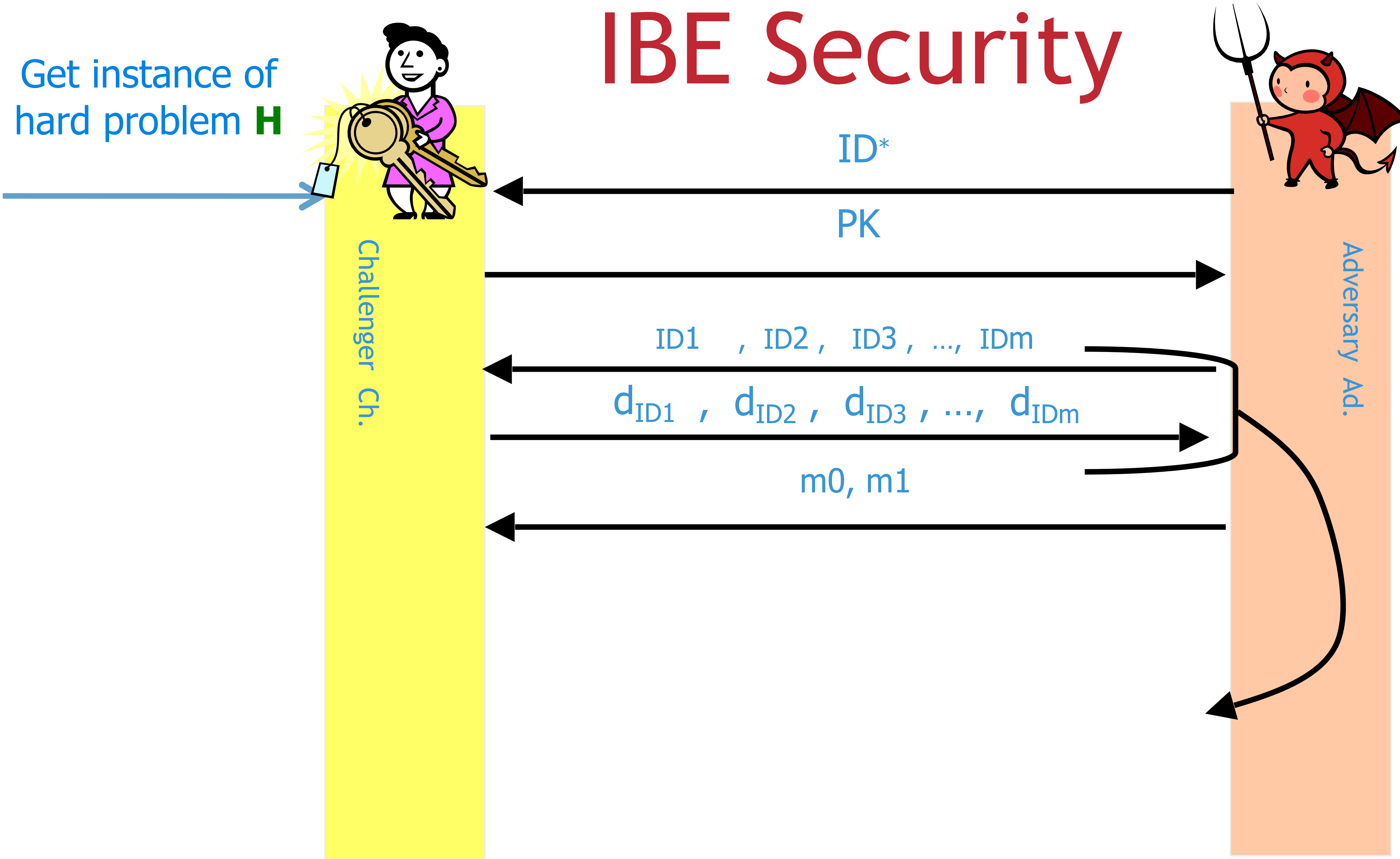

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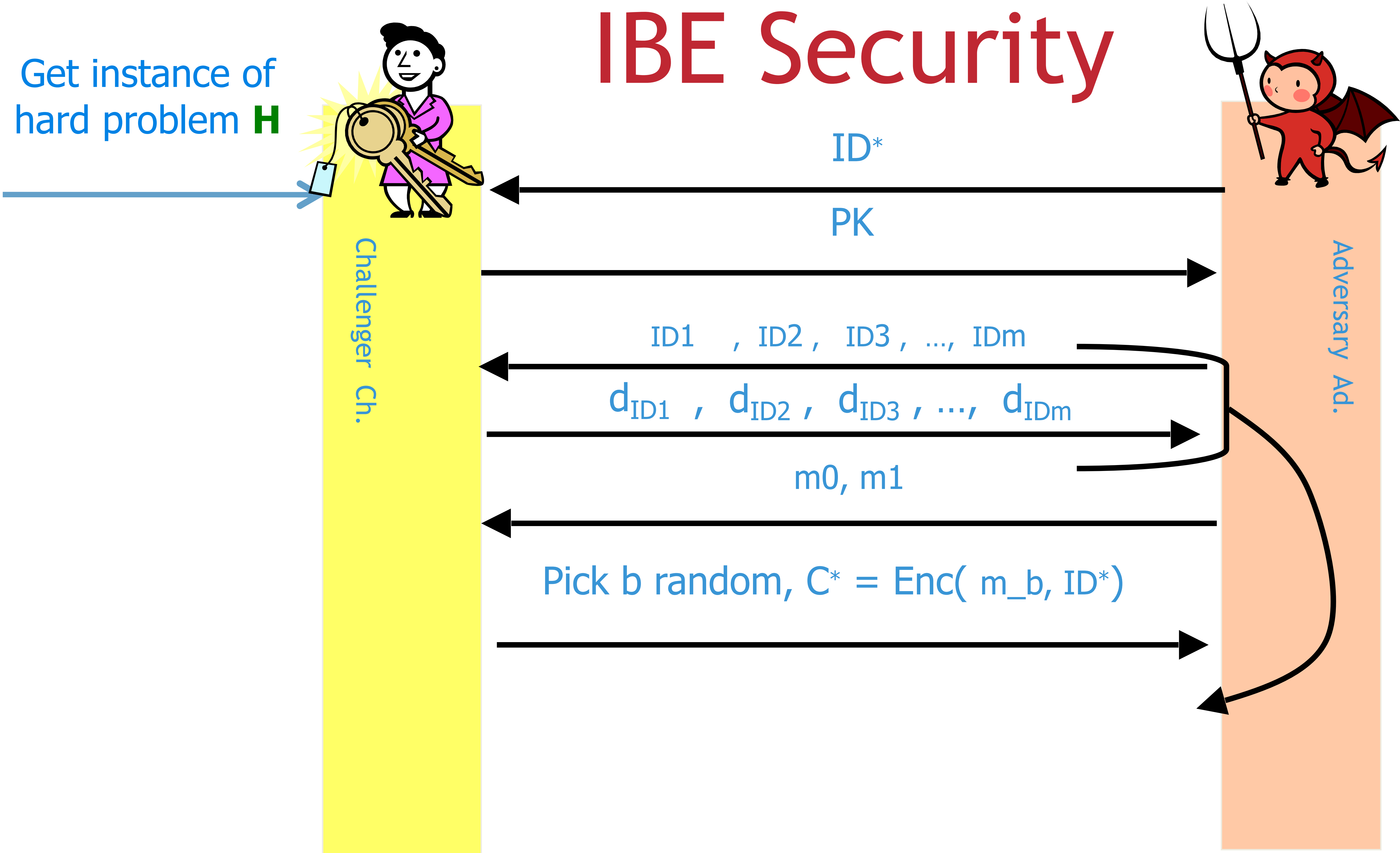

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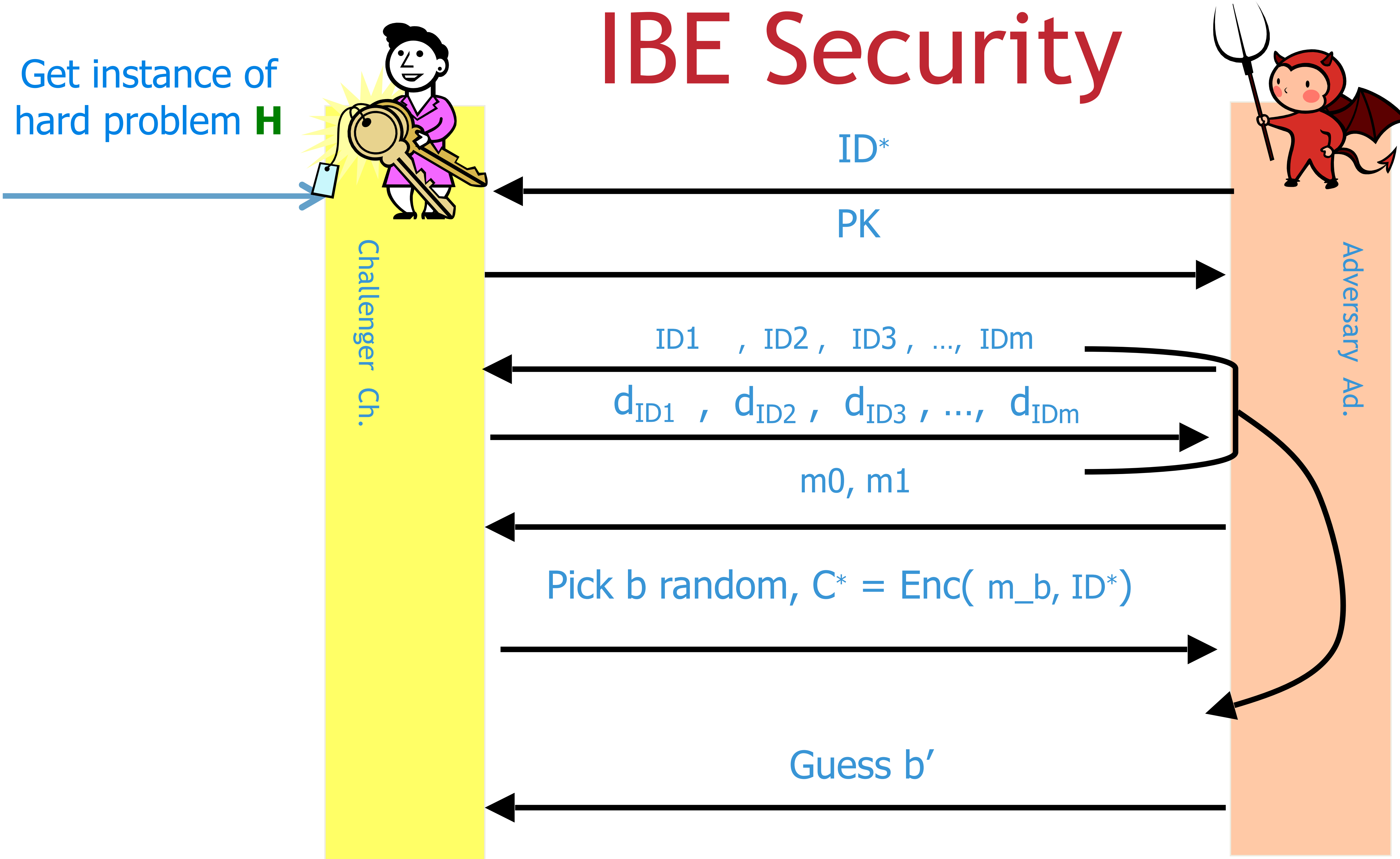

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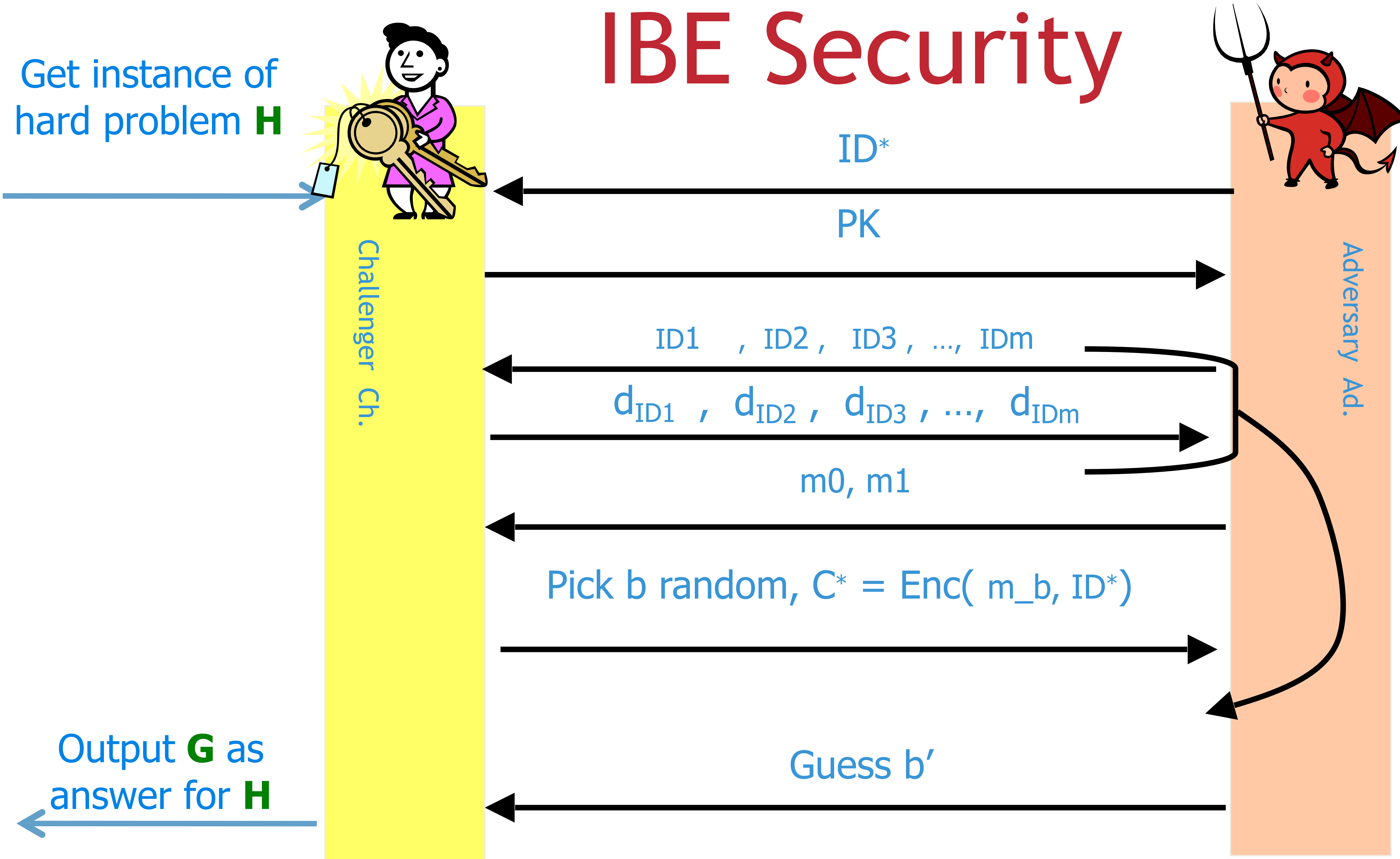

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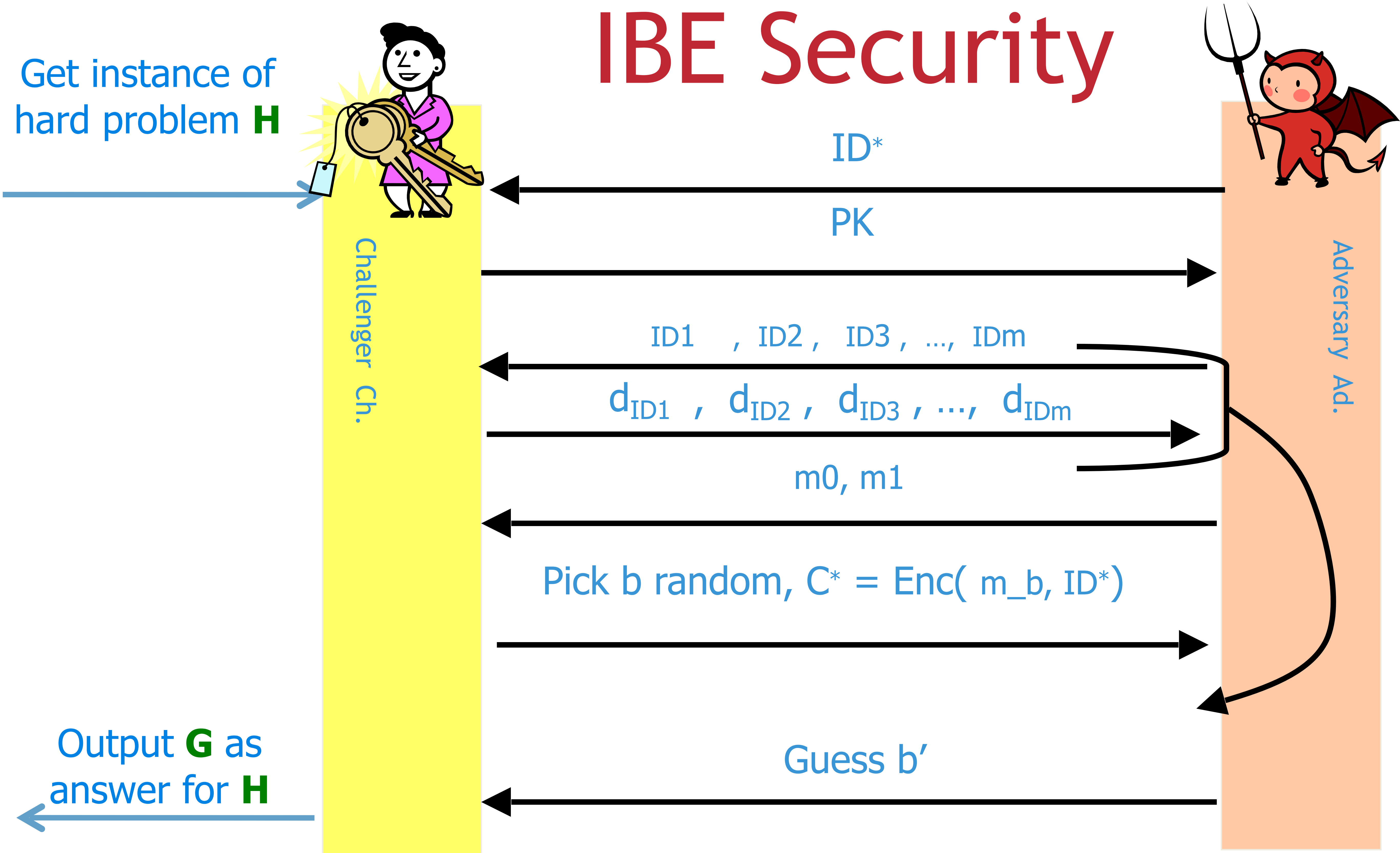

Attacker wins if | Pr[b=b'] - ½ | is non-negligible

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Small only if e is small

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- Proof in ROM allows to "program" H — gives exponential space to reduction!

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How to sample?

- Construction? Proof?

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- Its success translates to success for reduction/challenger!

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Know how to compute trapdoor for "extended" matrix $[A_0 \mid \textit{any}\,]$

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Encryption for id' = Regev PKE on matrix F_{id}

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Decryption : Regev decryption

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- ❖ $e^T C_1 = (F_{id}\, e)^T s$ + noise
- ❖ Since $F_{id}\, e = u \bmod q$, we have

Std Model Identity Based Encryption [ABB10]

$C_0 = u^T s + \text{noise} + m$ and $C_1 = F_{id}^T s + \text{noise}$

Decryption : Regev decryption

- Let $w = C_0 - e^T C_1$
- $e^T C_1 = (F_{id}\, e)^T s + \text{noise}$
- Since $F_{id}\, e = u \bmod q$, we have

$w = m + \text{noise}$ from which we can recover m.

Std Model Identity Based Encryption [ABB10]

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Simulation: Let challenge identity = id^*

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Random low norm matrix

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Simulation: Let challenge identity = id*

$F_{id} = [A_0 \mid A_1 + id\, G]$

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Std Model Identity Based Encryption [ABB10]

Let $\mathbf{B} \in \mathbb{Z}_q^{n\times m'}$, uniform $\mathbf{R} \in \mathbb{Z}_q^{m'\times n\log q}$, Gaussian

Let $\mathbf{A} =$

B	G - BR

Then, $\mathbf{A}$ uniform, admits LWE and SIS inversion

$$f_{\mathbf{A}}^{-1},\ g_{\mathbf{A}}^{-1}$$

Std Model Identity Based Encryption [ABB10]

MP12

Let $\mathbf{B} \in \mathbb{Z}_q^{n\times m'}$, uniform $\mathbf{R} \in \mathbb{Z}_q^{m'\times n\log q}$, Gaussian

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Let $\mathbf{B} \in \mathbb{Z}_q^{n\times m'}$, uniform $\mathbf{R} \in \mathbb{Z}_q^{m'\times n\log q}$, Gaussian

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- $F_{id} = [A_0 | A_0 R + (id - id^*) G]$
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Developed in ABB10

Std Model Identity Based Encryption [ABB10]

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Let $\mathbf{A} =$ | B | G - BR |

Then, $\mathbf{A}$ uniform, admits LWE and SIS inversion

$$f_{\mathbf{A}}^{-1}, \; g_{\mathbf{A}}^{-1}$$

Developed in ABB10

- $F_{id} = [A_0 | \; A_0 R + (id - id^*) G]$
- Can find basis for F_{id} given basis for G !
- Trapdoor vanishes for id = id*

Std Model Identity Based Encryption [ABB10]

Real System

Simulation

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System

Simulation

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System | Simulation

MSK = Trapdoor for A_0

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

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Simulation

MSK = R

Std Model Identity Based Encryption [ABB10]

$$\mathrm{PP} = A_0, A_1, G$$

Real System

MSK = Trapdoor for A_0

A_1 = Randomly chosen

Simulation

MSK = R

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System

MSK = Trapdoor for A_0

A_1 = Randomly chosen

Simulation

MSK = R

A_1 = $A_0R - id^* G$

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System		Simulation	
MSK	= Trapdoor for A_0	MSK	= R
A_1	= Randomly chosen	A_1	$= A_0R - id^* G$

Indistinguishable since R is random!

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System

MSK = Trapdoor for A_0

A_1 = Randomly chosen

Simulation

MSK = R

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Indistinguishable since R is random!

Encryption matrix $F_{id} = [A_0|A_1+id.G]$

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System		Simulation	
MSK	= Trapdoor for A_0	MSK	= R
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Real System

Encryption matrix $F_{id} = [A_0 | A_1 + id.G]$

Simulation

Encryption matrix $F_{id} = [A_0 \mid A_1 + id.G]$

$= [A_0 \mid A_0R + (id - id^*)G]$

Std Model Identity Based Encryption [ABB10]

$$PP = A_0, A_1, G$$

Real System

MSK = Trapdoor for A_0

A_1 = Randomly chosen

Encryption matrix $F_{id} = [A_0 | A_1 + id.G]$

Secret Key = short vector in F_{id}

Simulation

MSK = R

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MSK ➔ Key for any id

Simulation

MSK = R

$A_1 = A_0R - id^* G$

Encryption matrix $F_{id} = [A_0 \mid A_1 + id.G]$

$= [A_0 \mid A_0R + (id - id^*)G]$

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MSK ➔ Key for any id

Simulation

MSK = R

$A_1 = A_0R - id^* G$

Encryption matrix $F_{id} = [A_0 \mid A_1 + id.G]$

$= [A_0 \mid A_0R + (id - id^*)G]$

Secret Key = short vector in F_{id}

Trapdoor for G ➔ Key for id ≠ id*

Indistinguishable since R is random!

The matrix R

- Matrix R : each column randomly and independently chosen from $\{+1, -1\}^m$
- (A_0, A_1) indistinguishable from (A_0, A_0R)

  by leftover hash lemma

- Roughly states that R has enough entropy to make A_0R look like A_1

Generalizing to inner products (AFV11)

Generalizing to Inner Product (KSW08)

Key : $y = (y_1, \ldots, y_n)$

CT : $x = (x_1, \ldots, x_n)$

$$\text{Function } f(x, y) = \begin{cases} 1 & \text{If } \langle x \,.\, y\rangle = 0 \\ 0 & \text{otherwise} \end{cases}$$

Generalizing to Inner Product (KSW08)

Key : $y = (y_1, \ldots, y_n)$

CT : $x = (x_1, \ldots, x_n)$

Function $f(x, y)$ = 1 If $\langle x \cdot y \rangle = 0$

0 otherwise

Supports:

- OR –- Bob OR Alice $OR_{A,B}(z) = 1$ if $z = A \; OR \; z = B$

$$p(z) = (A - z)(B - z)$$

- CNF/DNF formulas of bounded size

Generalizing to Inner Product (KSW08)

Key : $y = (y_1, \ldots, y_n)$

CT : $x = (x_1, \ldots, x_n)$

Ciphertext Hides Attributes x_i

Function $f(x, y) = 1$ If $\langle x \,.\, y\rangle = 0$

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Supports:

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Generalizing to Inner Product (AFV11)

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- Parameters for |x| = |y| = 4:

$\{A_1\}\ \{A_2\}\ \{A_3\}\ \{A_4\} \qquad \{A\} \qquad u$

Generalizing to Inner Product (AFV11)

- Parameters for $|x| = |y| = 4$:

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- Master Secret Key: Trapdoor for A

Generalizing to Inner Product (AFV11)

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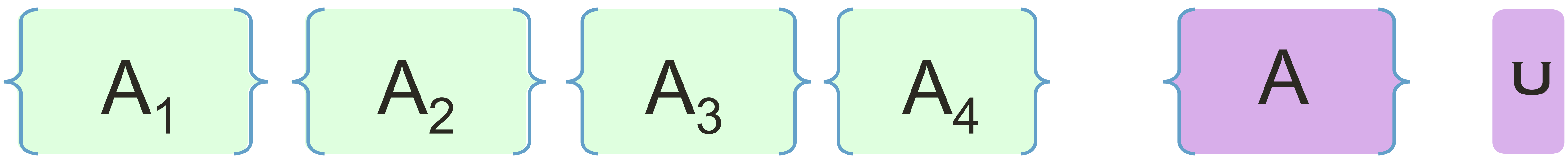

- Master Secret Key: Trapdoor for A
- Define $F_y = [A \mid \Sigma y_i A_i\,]$

Generalizing to Inner Product (AFV11)

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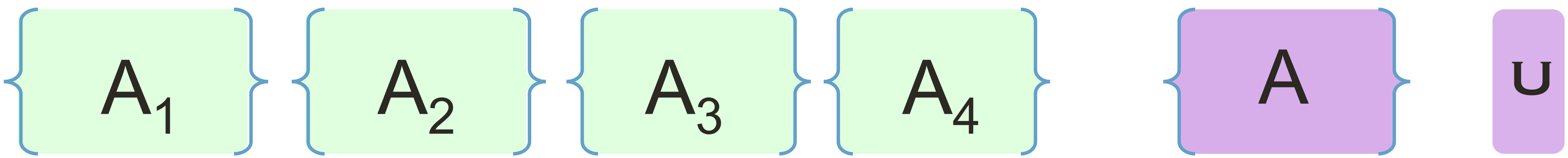

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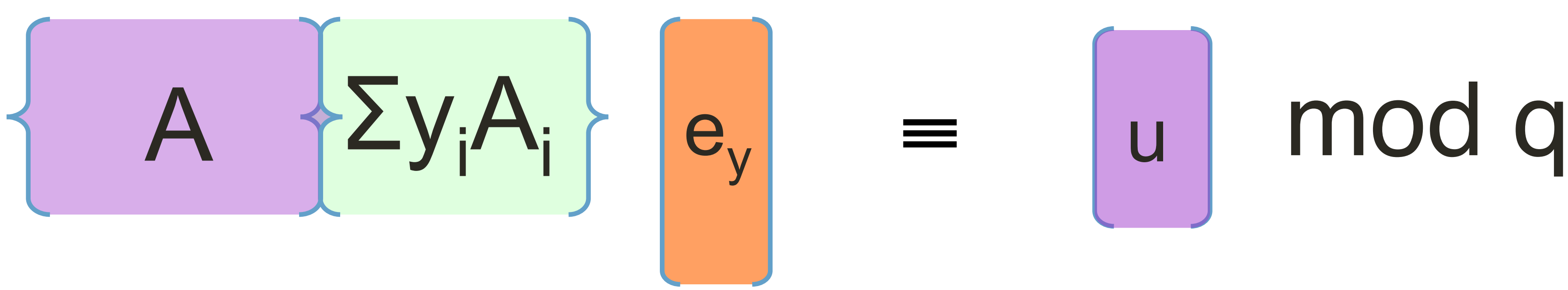

Generalizing to Inner Product (AFV11)

- Parameters for |x| = |y| = 4:

$$A_1 \quad A_2 \quad A_3 \quad A_4 \qquad A \qquad u$$

- Master Secret Key: Trapdoor for A

- Define $F_y = [A \mid \Sigma y_i A_i]$

$$\left[\, A \mid \Sigma y_i A_i \,\right] \, e_y \equiv u \mod q$$

key

Generalizing to Inner Product (AFV11)

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Encryption for vector $x = (x_1\ x_2\ x_3\ x_4)$:

Generalizing to Inner Product (AFV11)

Encryption for vector $x = (x_1\ x_2\ x_3\ x_4)$:

- ❖ Pick random vector s
- ❖ $C = u^T s + \text{noise} + \text{msg}$
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- ❖ Set $C_i = (A_i + x_i\ G)^T\ s + \text{noise}$

Generalizing to Inner Product (AFV11)

Decryption

(CT_x, SK_y) :

Generalizing to Inner Product (AFV11)

Decryption (CT_x, SK_y) :

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Generalizing to Inner Product (AFV11)

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Set $C_y = \Sigma\, y_i\, C_i$

$$= (\Sigma\, y_i\, A_i + \Sigma\, y_i\, x_i\, G)^T s + \Sigma\, y_i \text{ noise}$$

Generalizing to Inner Product (AFV11)

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$$[\, C' | C_y \,] = [A \mid \Sigma\, y_i\, A_i\,]^T s + \text{noise}$$

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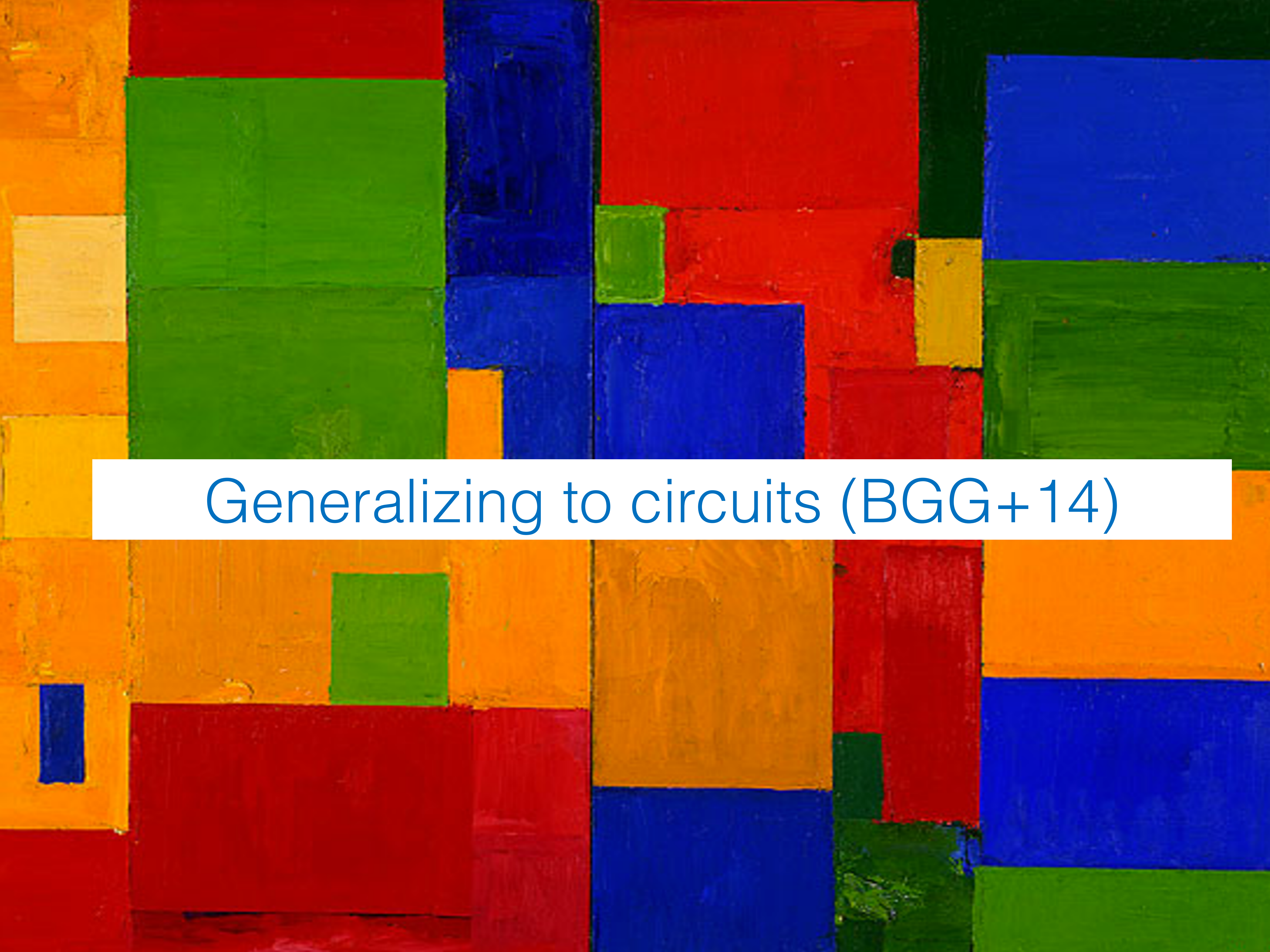
But this is what we have the key for !

Perform Regev Decryption.



Generalizing to circuits (BGG+14)

Recall Ciphertext Structure

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Encryption for vector $x = (x_1\ x_2\ x_3\ x_4)$:

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Previously: Could evaluate on CT to obtain

$C_{<x,\, y>} = (A_y + \langle x, y \rangle\, G)^T s + \text{noise}$

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When <x, y> = 0, obtain CT that encodes f alone,
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Previously: Could evaluate on CT to obtain

$C_{\langle x, y\rangle} = (A_y + \langle x, y\rangle\, G)^T s + \text{noise}$

When <x, y> = 0, obtain CT that encodes f alone,
Keygen may compute matching key

Generalize to arbitrary f?

$C_{f(x)} = (A_f + f(x)\, G)^T s + \text{noise}$

Handling Multiplication [BGG+14]

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$C_1 = (A_1 + x_1\, G)^T\, s + \text{noise}$

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Want $\quad C_{x1\ x2} = (A_{12} + x_1 x_2\, G)^T s + \text{noise}$

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$(A_1 + x_1 G)$

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Want $\quad C_{x1\ x2} = (A_{12} + x_1 x_2\, G)^T s + \text{noise}$

Key Observation: x may be used in evaluation !

$(A_1 + x_1\, G)\ G^{-1}\,(-A_2)$

$(A_2 + x_2\, G)$

Handling Multiplication [BGG+14]

Recall $G\, G^{-1}(A) = A$

$C_1 = (A_1 + x_1 G)^T s + \text{noise}$ $\quad$ $C_2 = (A_2 + x_2 G)^T s + \text{noise}$

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$(A_1 + x_1 G)\ G^{-1}(-A_2)$

$(A_2 + x_2 G)\ (x_1)$

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$(A_2 + x_2 G)\ (x_1)$

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More Generally [BGG+14]. . .

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Perform Regev Decryption as usual

Generalizes to all circuits [BGG+14]

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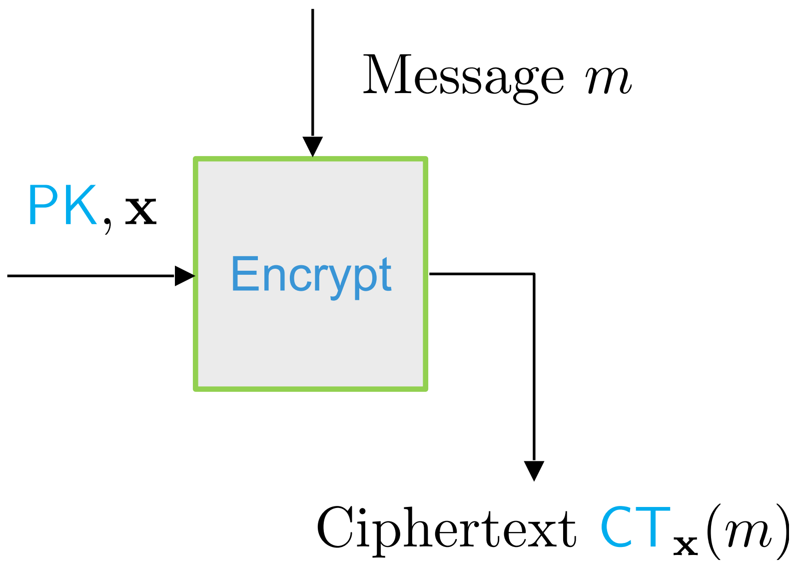

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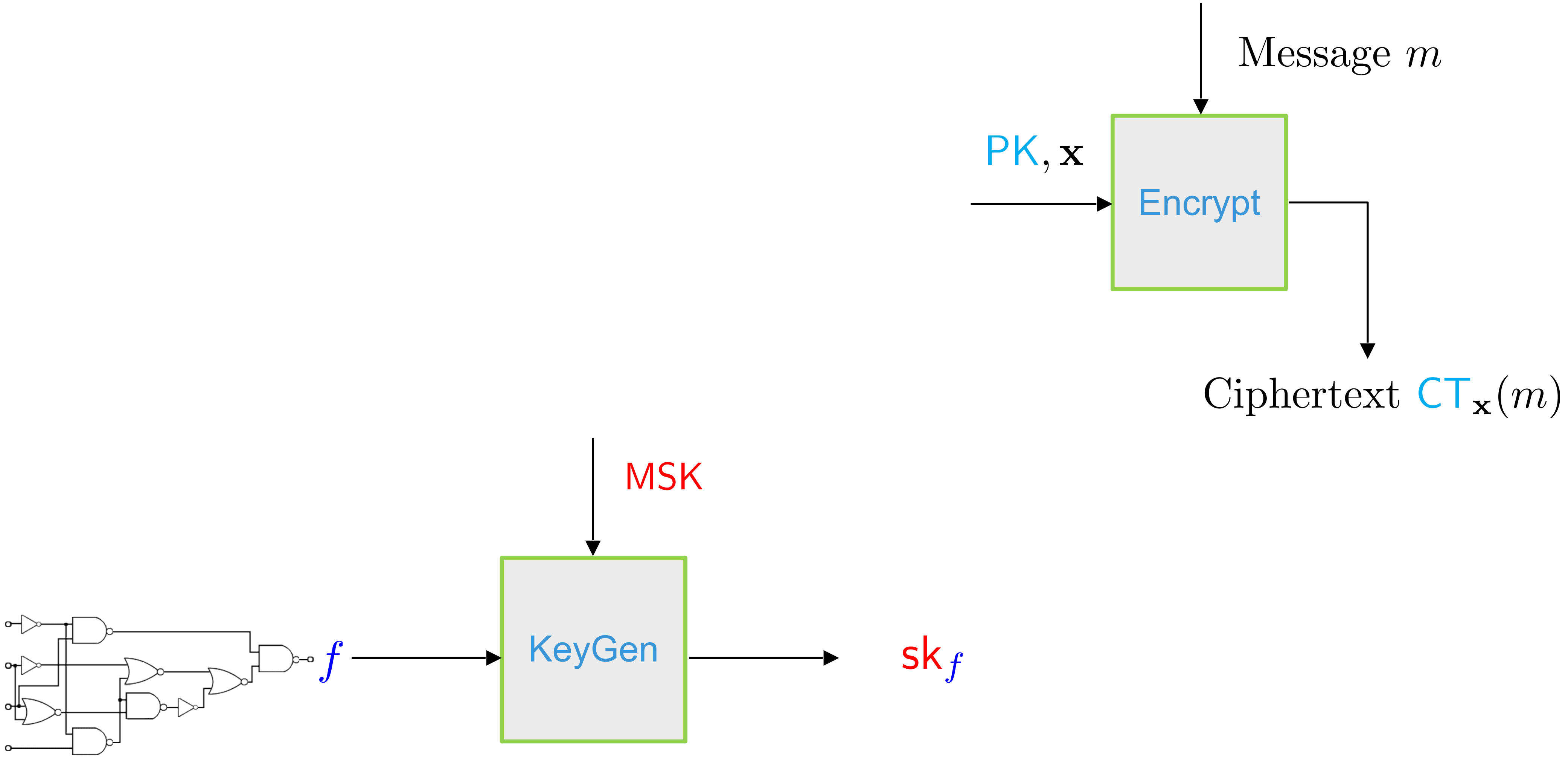

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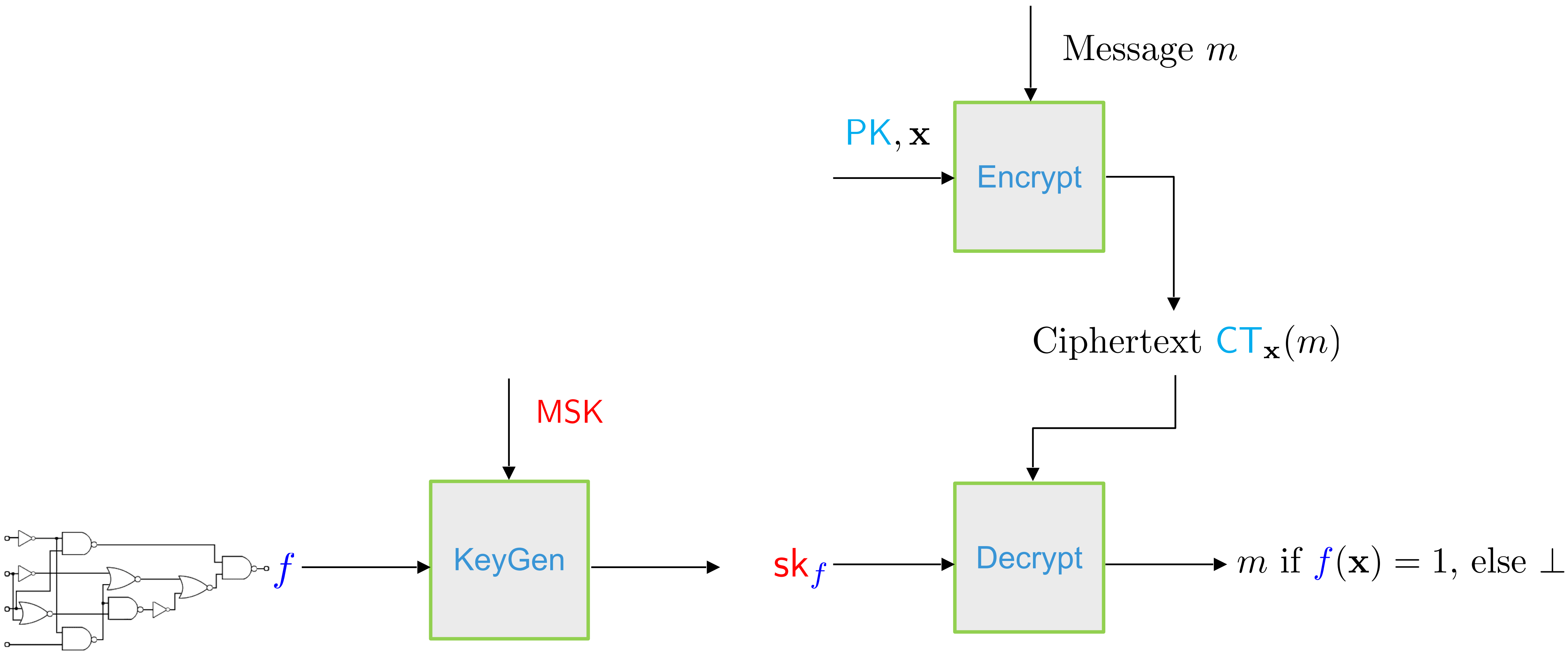

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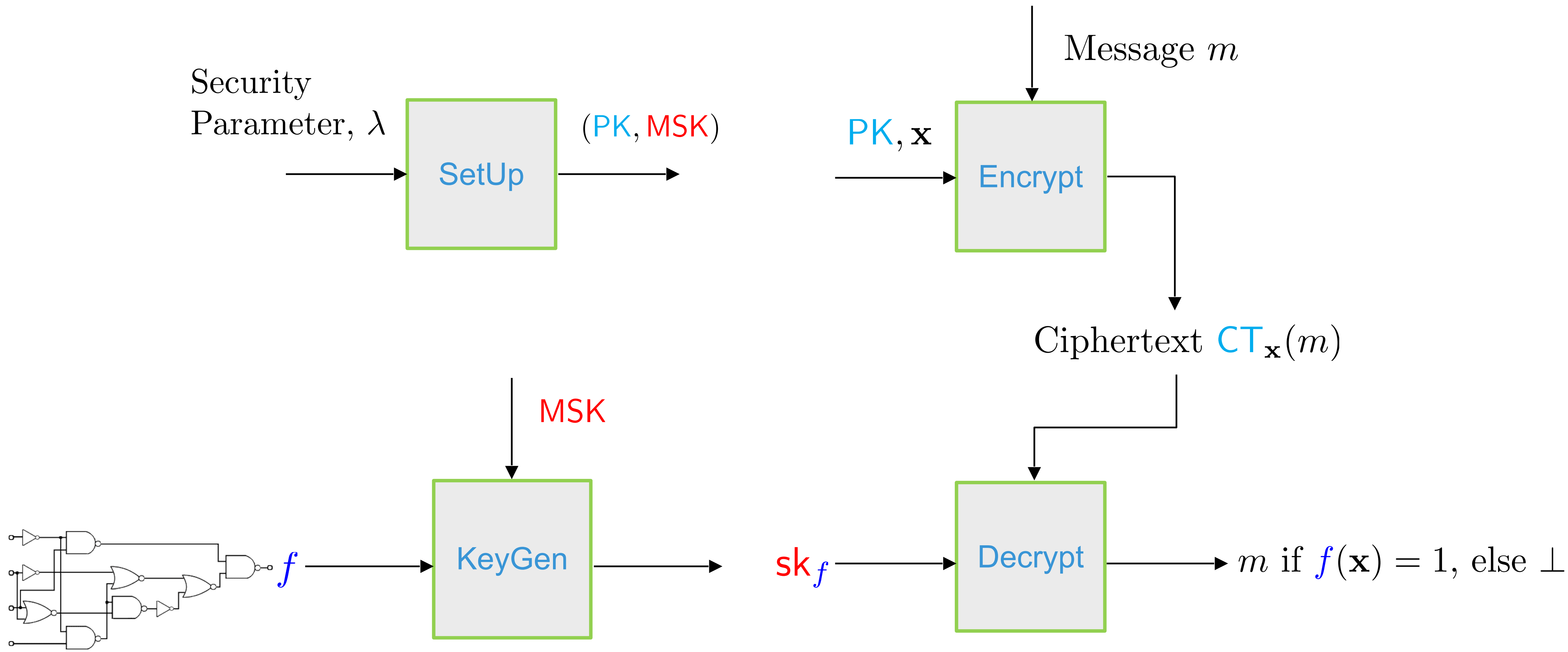

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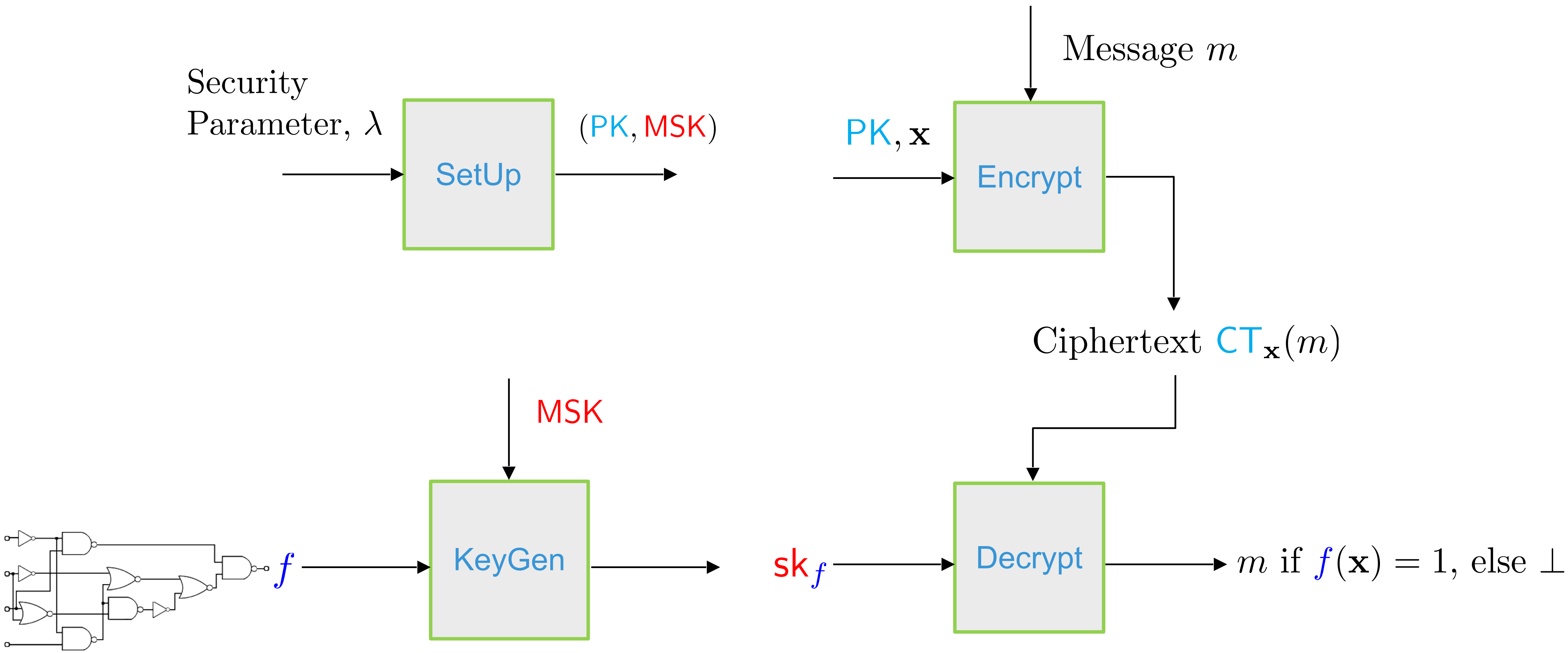
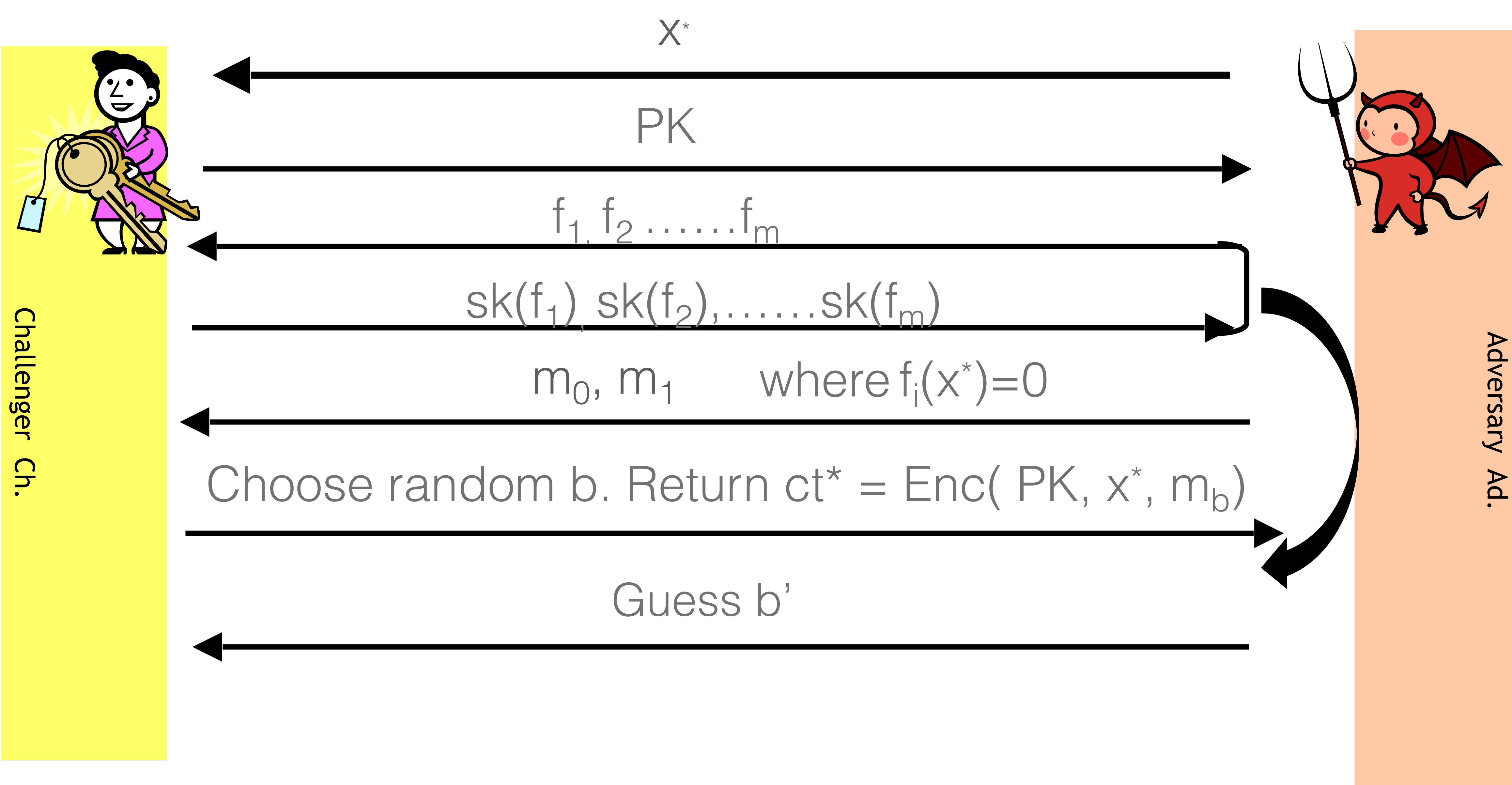


Attribute based Encryption (ABE) [SW05]

Security Definition



Attacker wins if | Pr[b=b'] - ½ | is non-negligible

Security: Challenges

- Challenger needs to be able to answer private key queries of Adversary: much more complex!
- Challenger can't have master trapdoor(Trapdoor for A)
- Must embed LWE challenge into challenge ciphertext

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Strategy: Key Queries

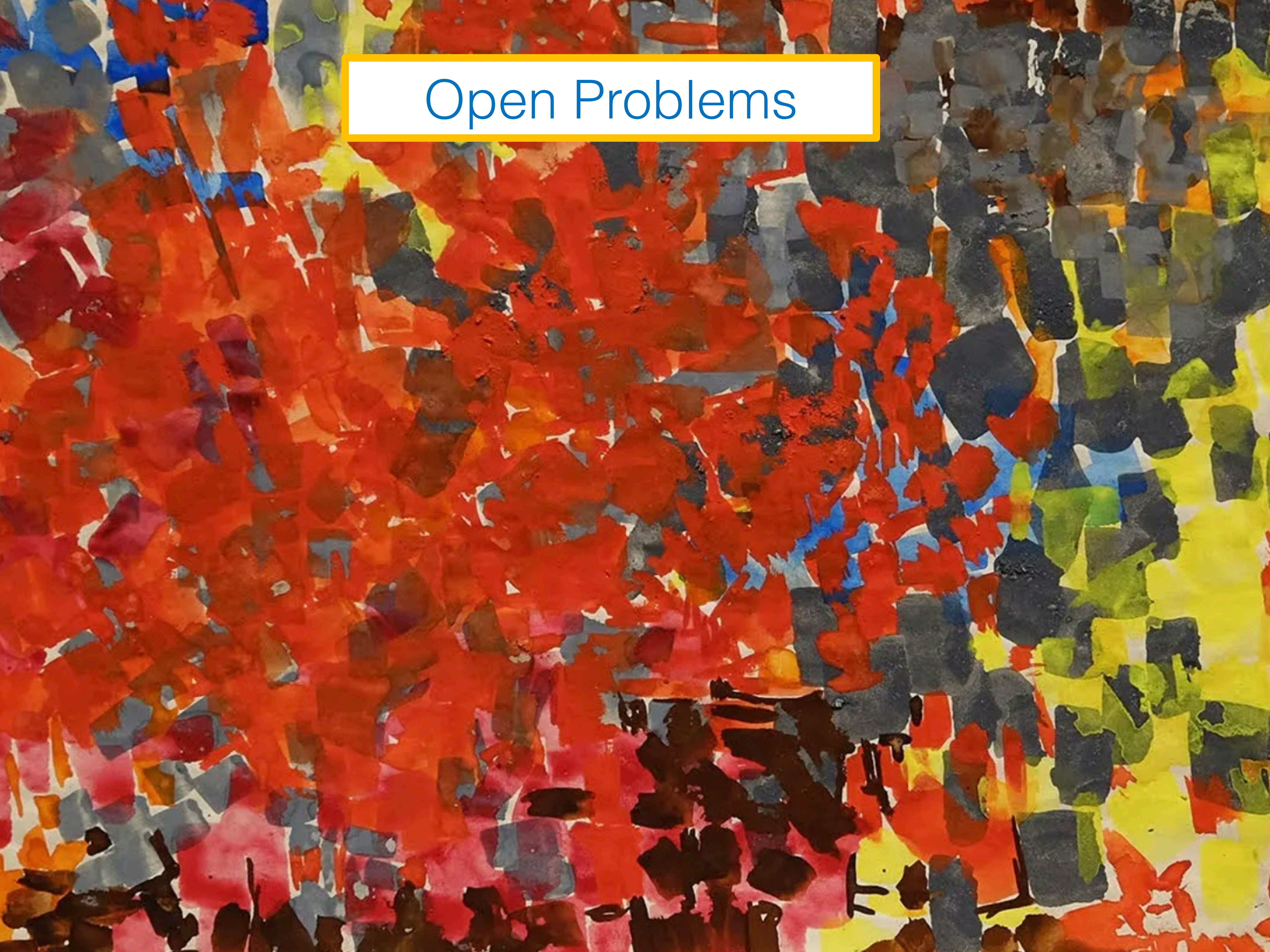
- Need TD for $[A \mid A_f]$ when $f(x^*) \neq 0$.
- $A_f = [AR_f - f(x^*)G]$. Let $H = f(x^*)$.
- Recall

Let $\mathbf{A} \in \mathbb{Z}_q^{n \times m'}$ uniform, $\mathbf{R} \in \mathbb{Z}_q^{m' \times n \log q}$ small

Then $[\, A \mid AR - H\, G \,]$

admits LWE and SIS inversion.

Open Problems



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Thank You!

Image Credits : Hans Hoffman, Joan Mitchell, Lynn Drexler