Introduction to Lattices

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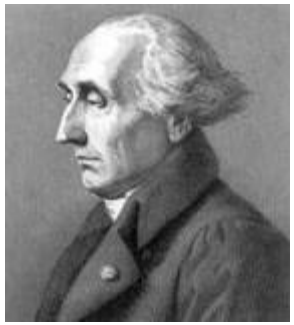
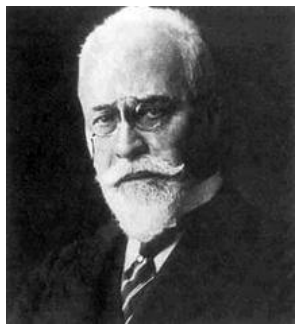
(Point) Lattices

- Traditional area of mathematics
  - Bridge between number theory and geometry
  - Studied by Lagrange, Gauss, ..., Minkowski, ...





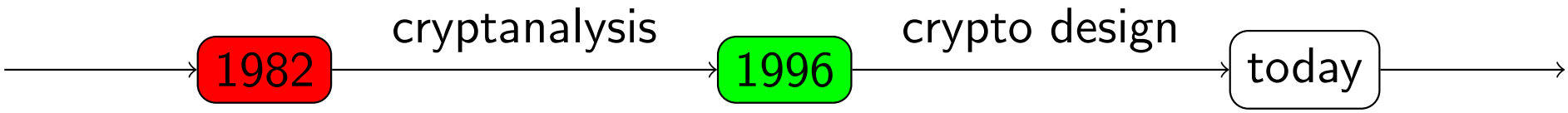
- Many applications in computer science and cryptography
  - Cryptanalysis: breaking low-exponent RSA
  - Coding Theory: error correcting codes for wireless communication
  - Optimization: Integer Programming
  - Cryptography: post-quantum cryptography and much more

Outline

(Post-quantum) Lattice-based Cryptography

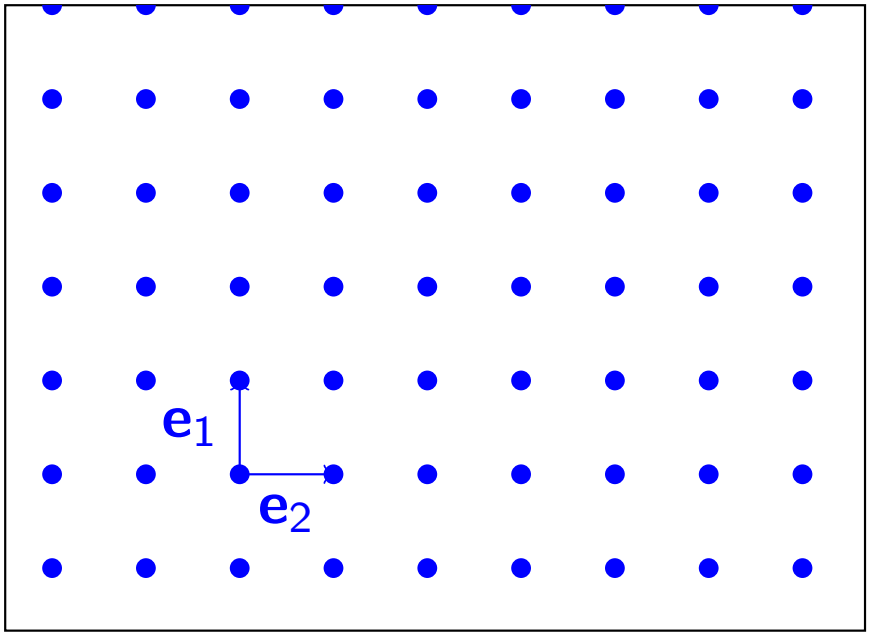
- Post-quantum cryptography
  - Can be used on conventional (non-quantum) computers
  - Remain secure in the face of quantum attacks
- Lattice problems believed to be hard even for quantum computers
- NIST Post-Quantum Cryptography standardization
  - Process started in 2016
  - A good half of the ~ 80 submissions based on lattices
  - 2023: lattice-based encryption and signatures picked as PQC standard
- Many other desirable properties of lattice-based cryptography
  - Fast and Parallelizable
  - Versatile: many advanced applications
  - Only known solution for some: Fully Homomorphic Encryption

Lattice Cryptography: a Timeline



- Lenstra, Lenstra, Lovasz (1982) : The "LLL" algorithm
- Ajtai (1996) : Hardness of "Short Integer Solution" (SIS) problem
  - Ajtai, Dwork (1997): Pubic Key Encryption
  - Hoffstein, Pipher, Silverman (1998): NTRU cryptosystem
- M. (2002) : "Generalized compact knapsacks" (RingSIS)
  - Efficient version of Ajtai's construction, based on structured lattices
- Regev (2005) : Hardness of "Learning with Errors" (LWE)
  - Injective variant of Ajtai's SIS, with wider range of applications
- ... RingLWE ... Fully Homomorphic Encryption ... Lattice Trapdoors
- 2016-2023: NIST Post-Quantum Cryptography Standardization

Lattices: Definition



The simplest lattice in n-dimensional space is the integer lattice

$$\Lambda = \mathbb{Z}^n$$

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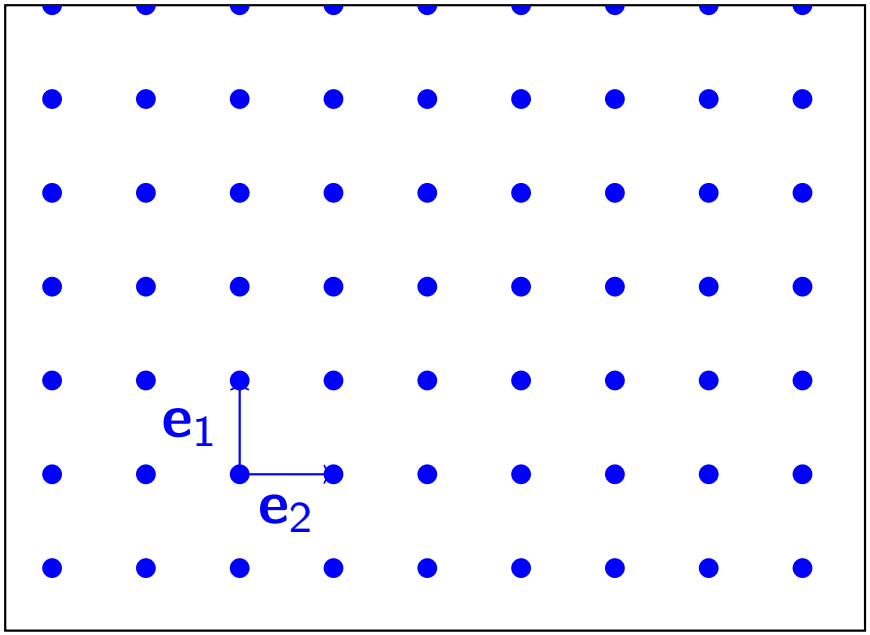

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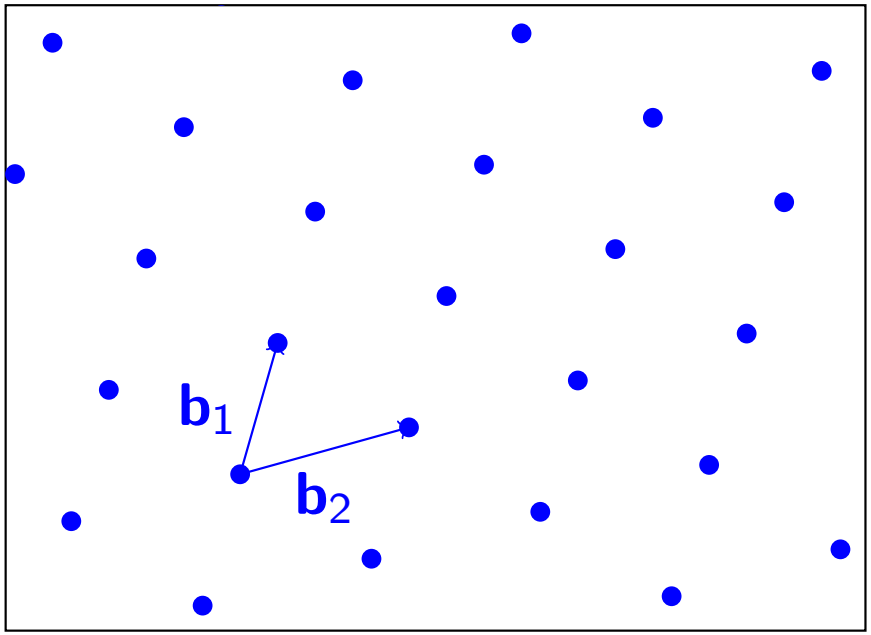


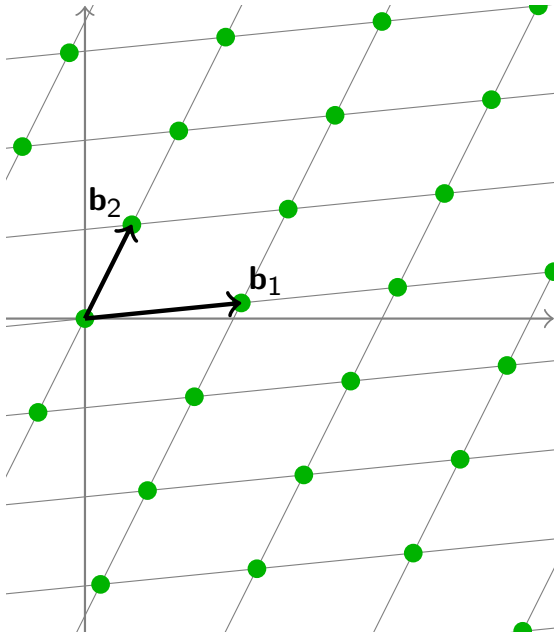
Other lattices are obtained by applying a linear transformation

$$\Lambda = \mathbf{B}\mathbb{Z}^n \qquad (\mathbf{B} \in \mathbb{R}^{d \times n})$$

Lattices and Bases

A lattice is the set of all integer linear combinations of (linearly independent) basis vectors $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{R}^n$:

$$\mathcal{L} = \sum_{i=1}^{n} \mathbf{b}_i \cdot \mathbb{Z}$$



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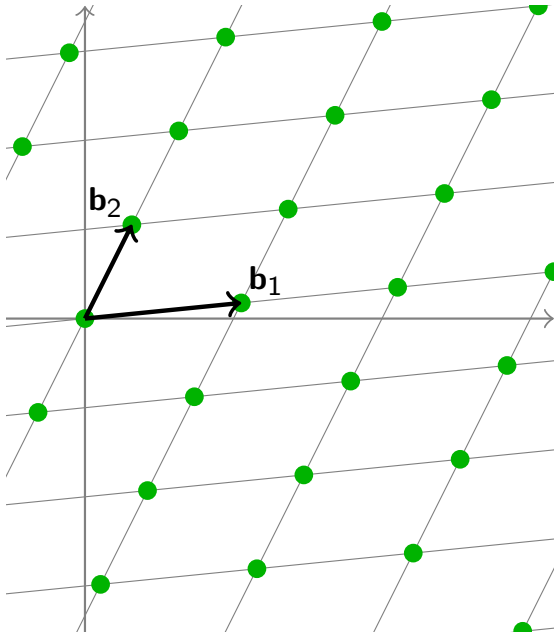

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The same lattice has many bases

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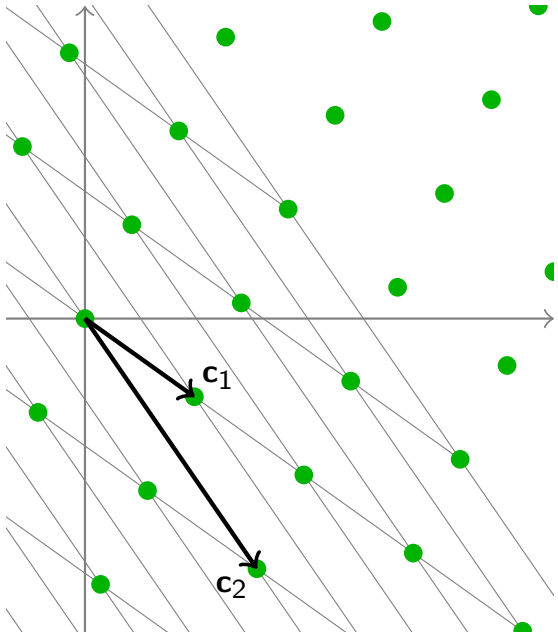

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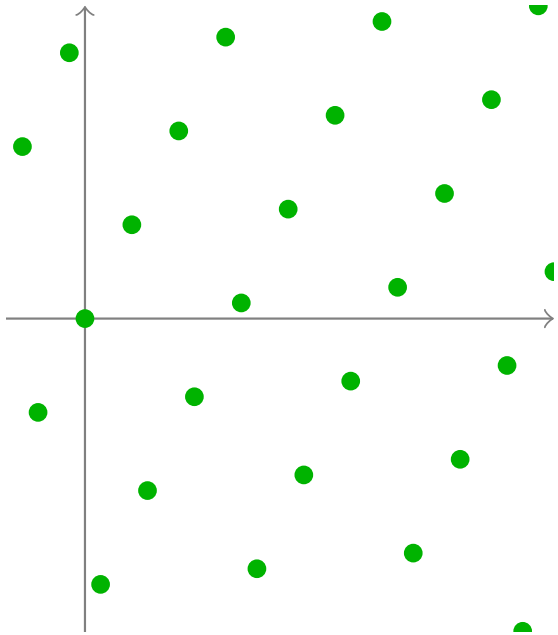


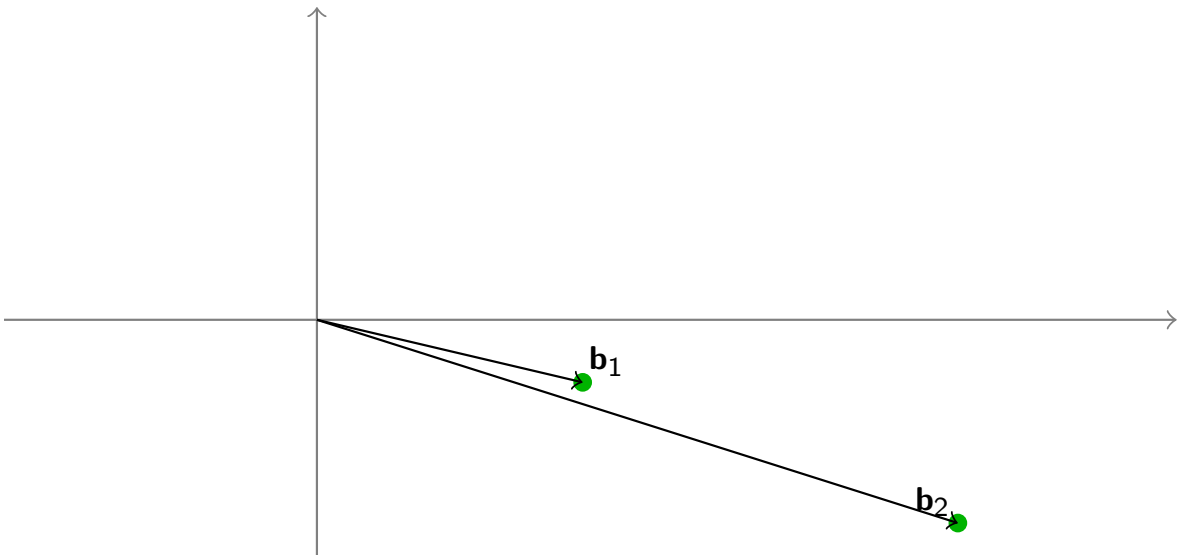
Definition (Lattice)

A discrete additive subgroup of $\mathbb{R}^n$

Shortest Vector Problem

Definition (Shortest Vector Problem, SVP)

Given a lattice $\mathcal{L}(\mathbf{B})$, find a (nonzero) lattice vector $\mathbf{Bx}$ (with $\mathbf{x} \in \mathbb{Z}^k$) of length (at most) $\|\mathbf{Bx}\| \leq \lambda_1$



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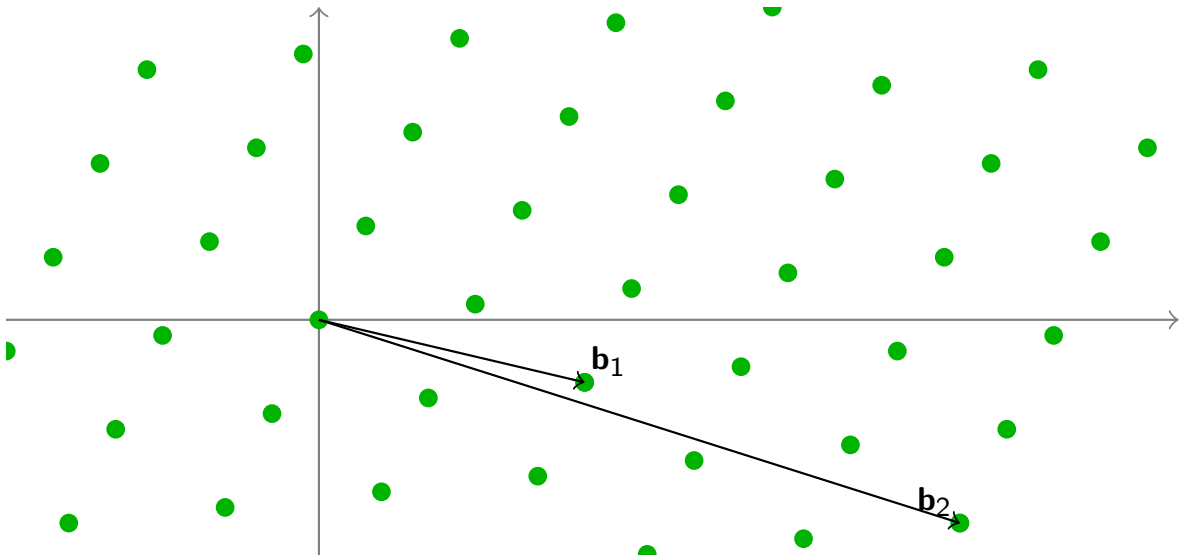

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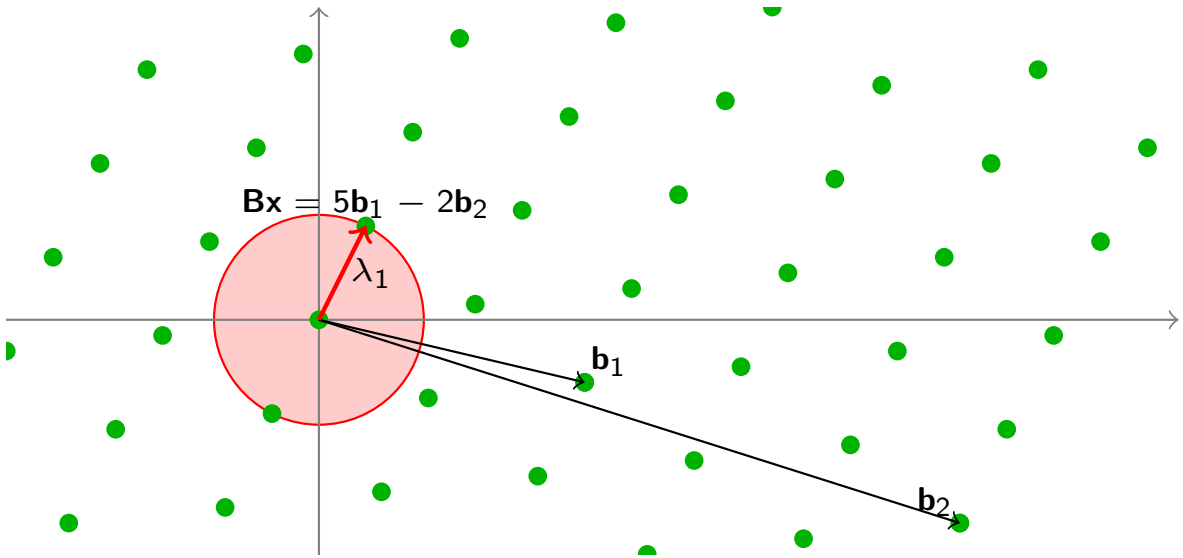

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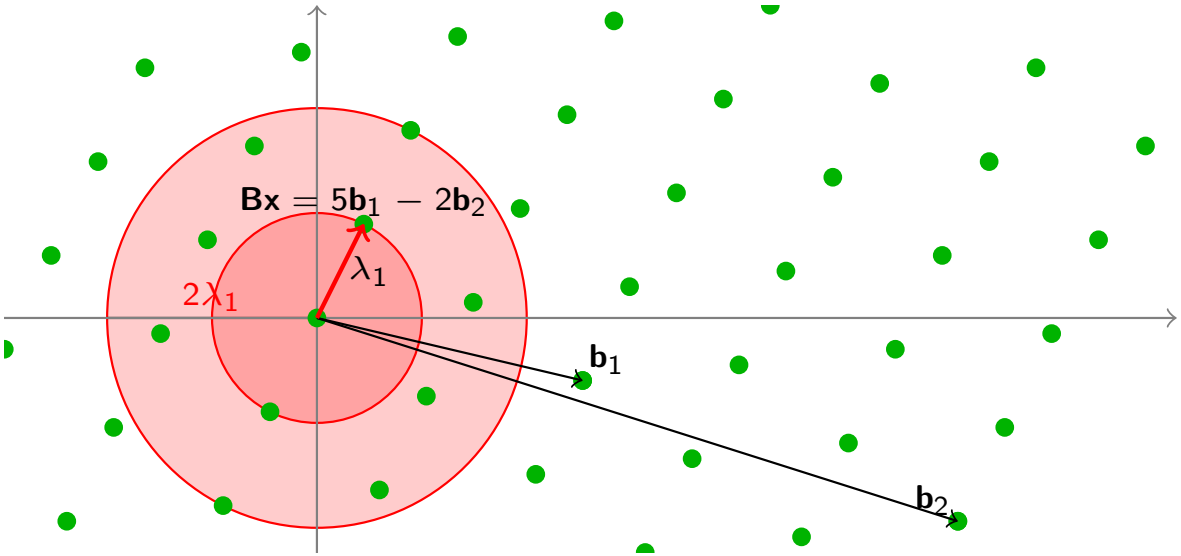

Closest Vector Problem

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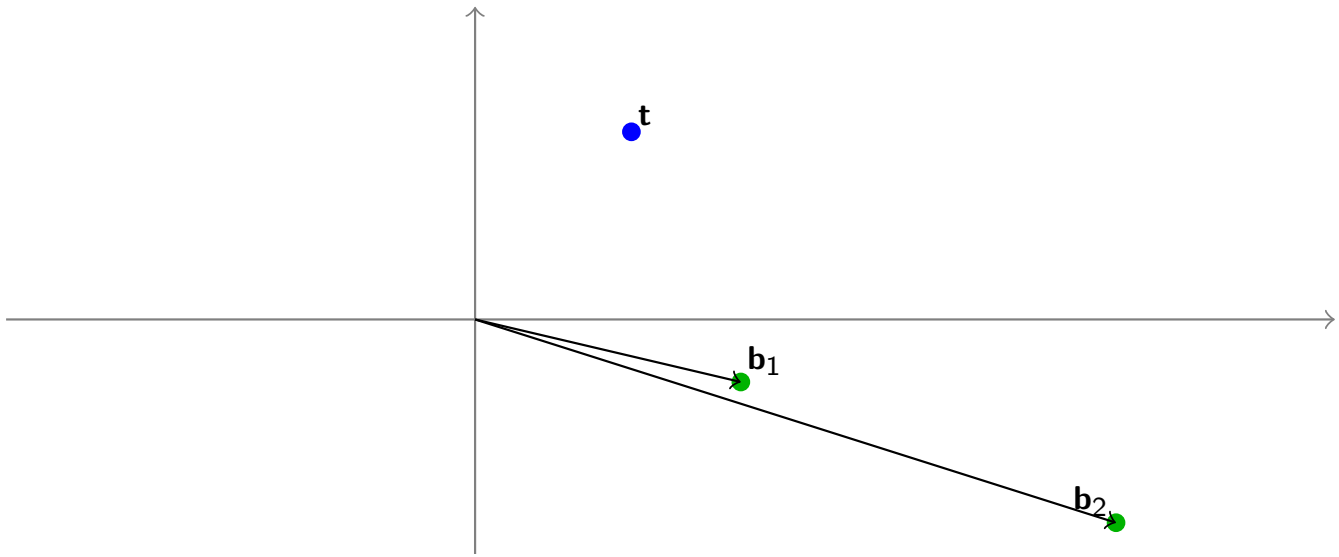
Given a lattice $\mathcal{L}(\mathbf{B})$ and a target point $\mathbf{t}$, find a lattice vector $\mathbf{Bx}$ within distance $\|\mathbf{Bx} - \mathbf{t}\| \leq \mu$ from the target



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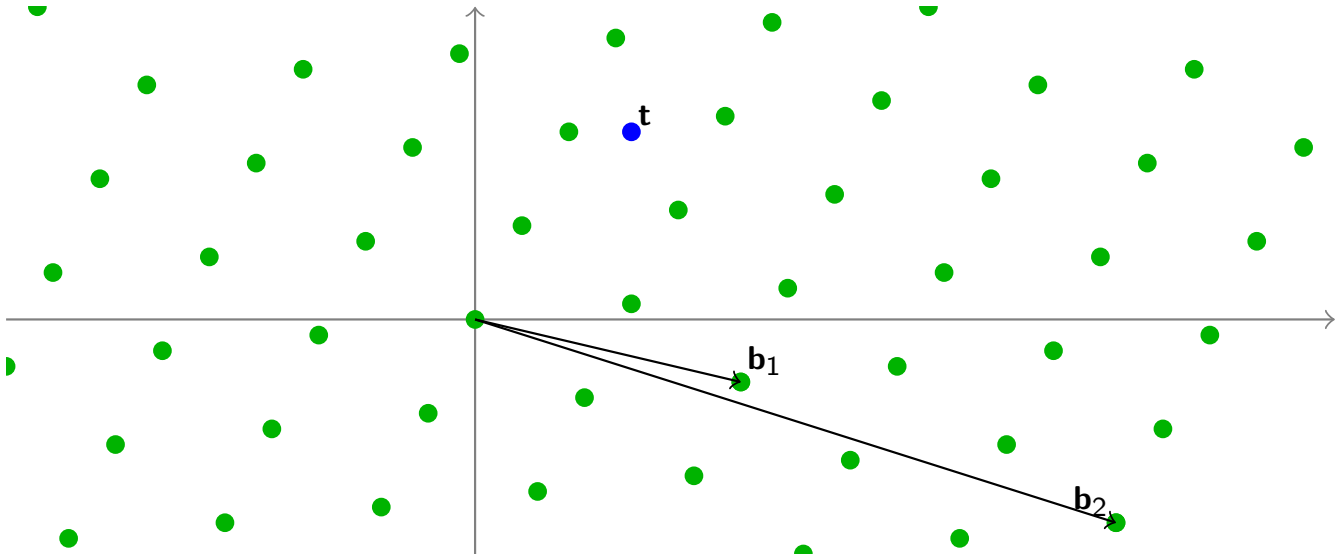

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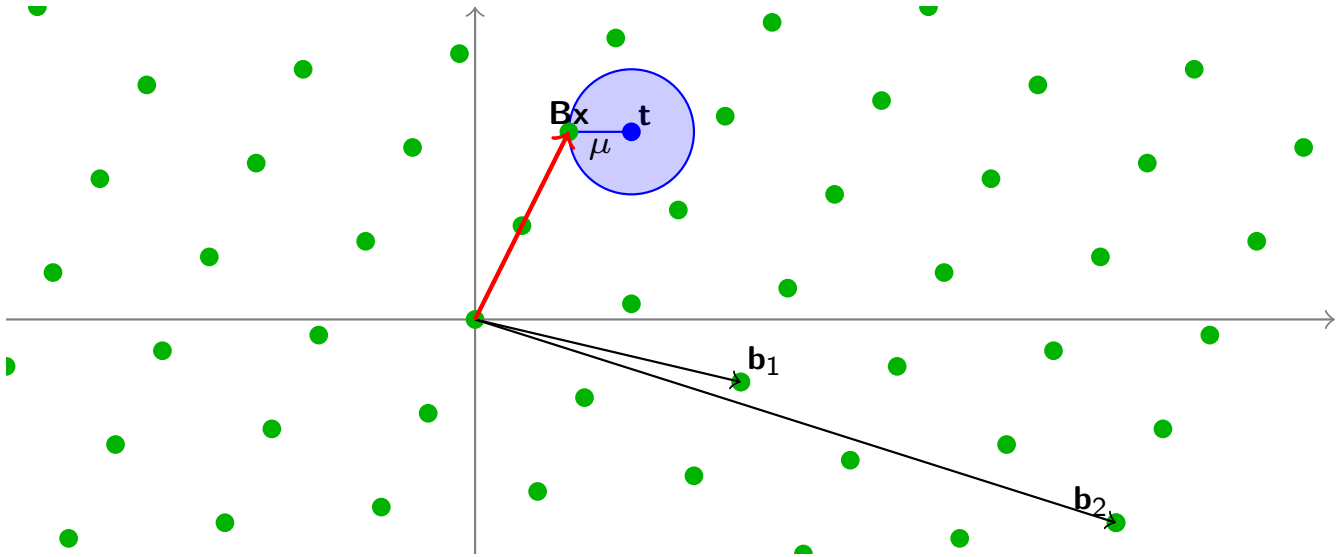

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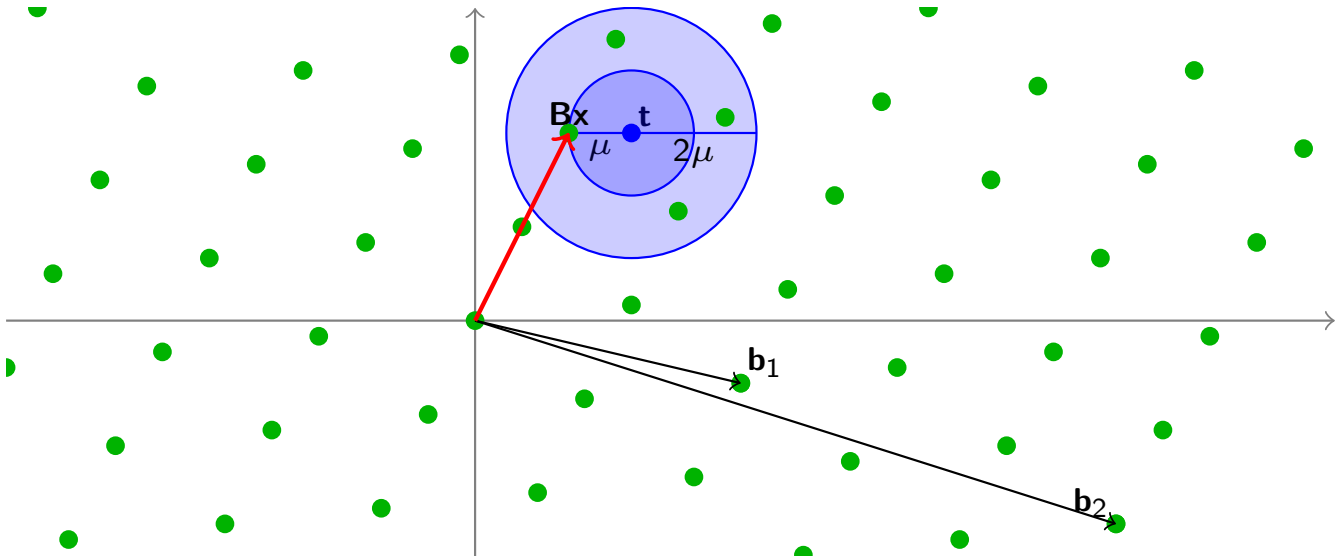

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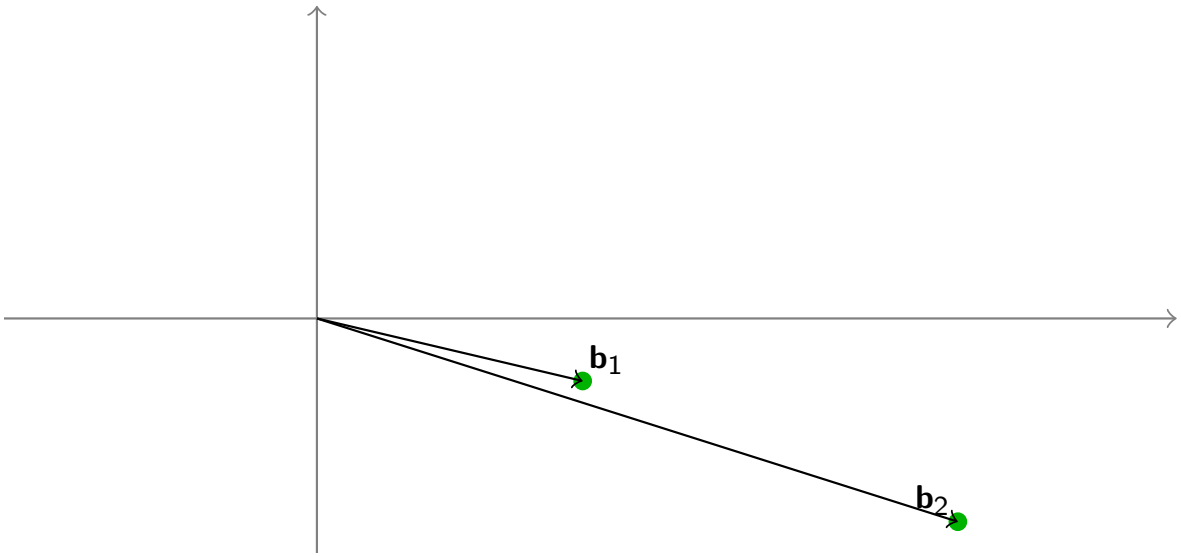

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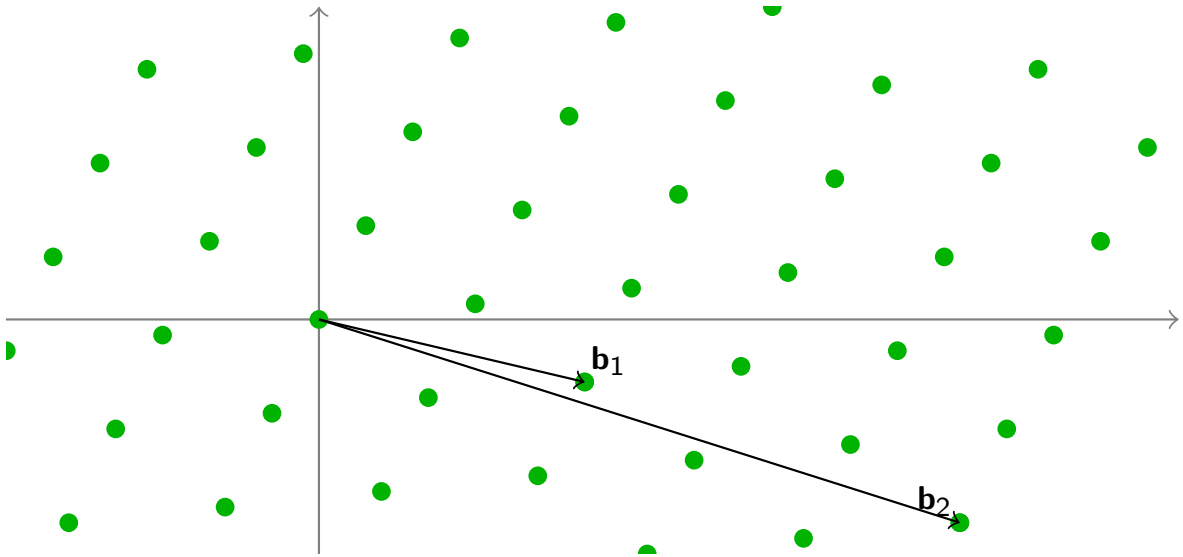

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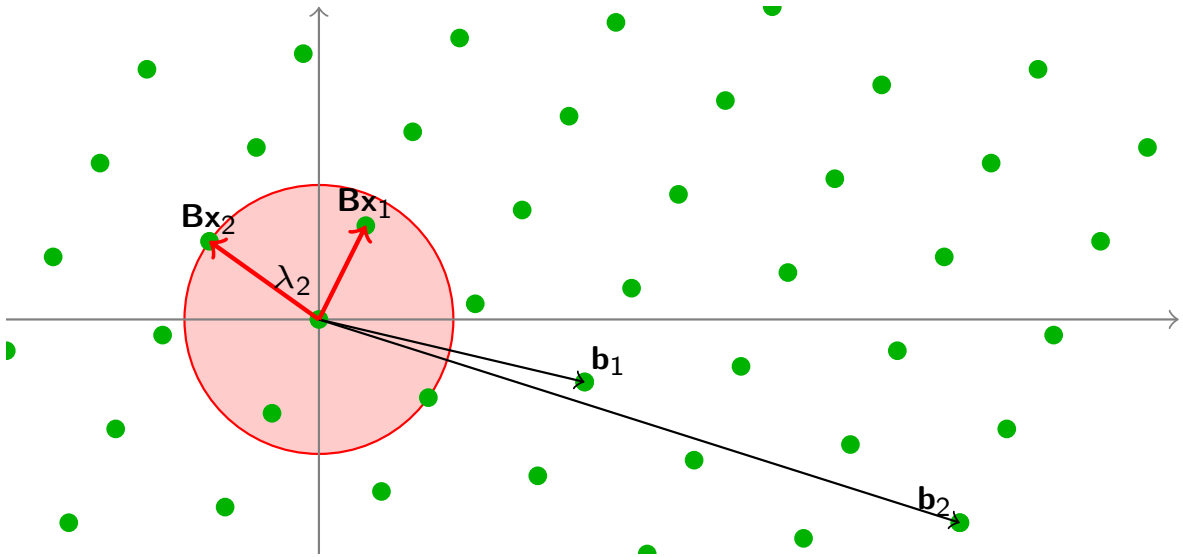

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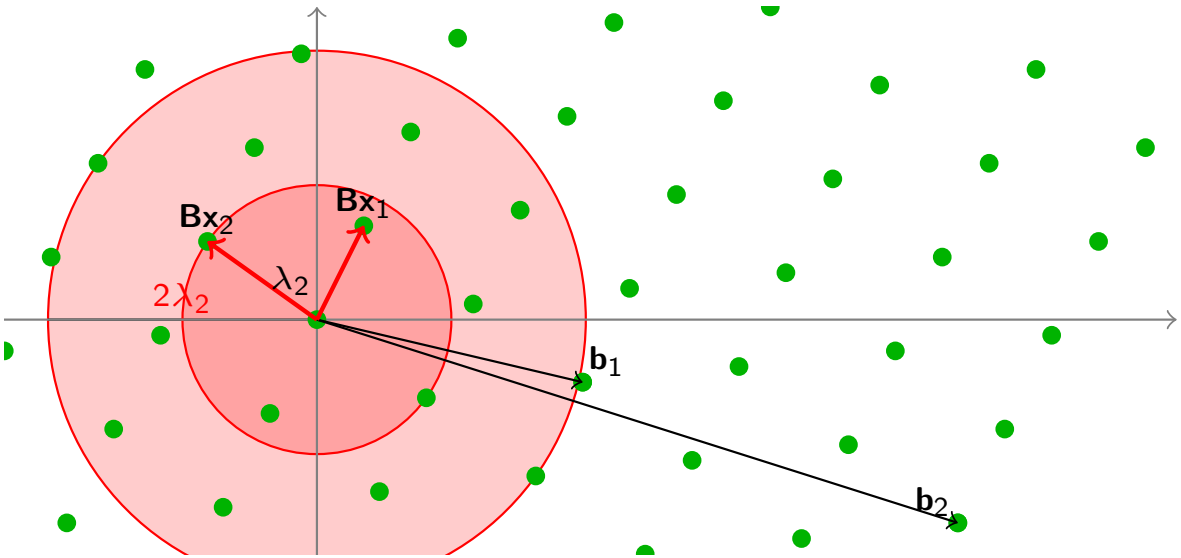
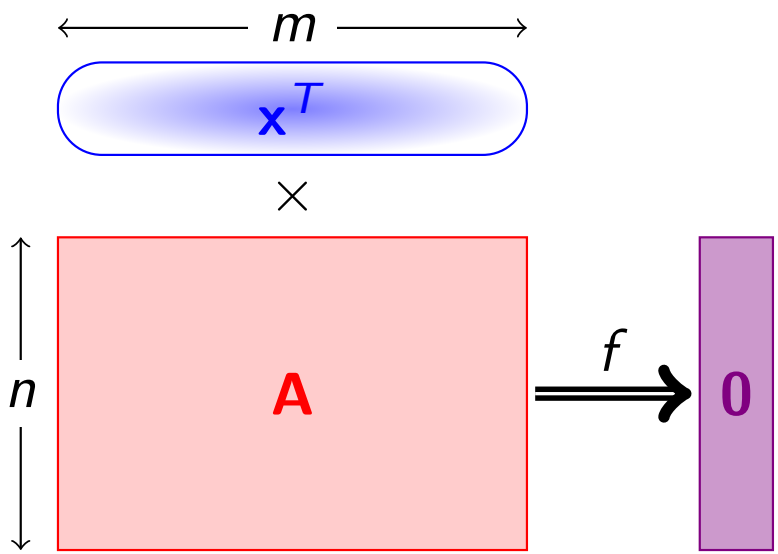


Short Integer Solution (SIS): random lattices

- Parameters: $m, n, q \in \mathbb{Z}$
- Key: $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$
- Function: $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$
- Choose $\mathbf{A}$ uniformly at random

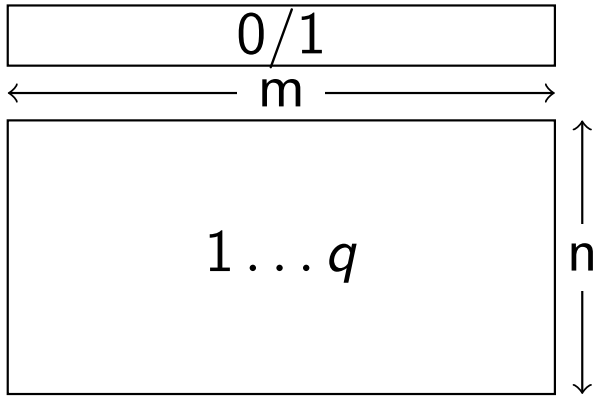


The SIS lattice: $\Lambda_q^{\perp}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{Z}^m : \mathbf{A}\mathbf{x} = \mathbf{0} \pmod q\} \subseteq \mathbb{Z}^m$.

- Does $\Lambda_q^{\perp}(\mathbf{A})$ contain short nonzero vectors?
- Can you solve SVP in $\Lambda_q^{\perp}(\mathbf{A})$ when $\mathbf{A}$ is chosen at random?
  - If not, what makes it hard?
  - If not, how is it useful for cryptography?

SIS parameters and Collision Resistant Hashing

- $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$ where $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$
- Restrict $f_{\mathbf{A}}$ to $\mathbf{x} \in \{0,1\}^m$
- Parameters:
  - n: main security parameter
  - $q = n^2 = n^{O(1)}$ small modulus
  - $m = 2n \log_2 q = O(n \log n)$
  - e.g., $n = 256$, $q = 2^{16}$, $m = 8192$



- $f_{\mathbf{A}}$ is a compression function mapping $8192 \to 4096$ bits
  - There exist collisions $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} \pmod q$
  - Equivalently, $\mathbf{z} = \mathbf{x} - \mathbf{y} \in \{0, 1, -1\}^m$ satisfies $\mathbf{A}\mathbf{z} = \mathbf{0} \pmod q$
  - $\mathbf{z} \in \Lambda_q^{\perp}(\mathbf{A})$ is a short nonzero lattice vector of length $\|\mathbf{z}\|_\infty \leq 1$

Remark

If SVP on $\Lambda_q^{\perp}(\mathbf{A})$ is hard, then $f_{\mathbf{A}}$ is a collision resistant hash function!

Example: Efficiency

- Function: $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$ where $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$
- Example parameters: $n = 2^8$, $q = 2^{16}$, $m = 2n \log_2 q = 2^{13}$
- $f_{\mathbf{A}}$ maps $\mathbf{x} \in \{0,1\}^m$ (8192 bits) to $\mathbb{Z}_q^n$ (4096 bits)
- Computing $f_{\mathbf{A}}$ takes $n \cdot m = 2^{21}$ additions and multiplications on small (2-byte) numbers
  - Easy to parallelize: using AVX512 SIMD instructions at 4 GHz, this is just $32\mu s$.
- Key Storage: $\mathbf{A}$ takes $n \cdot m = 2^{22}$ bytes, or $4MB$
  - Storage is not very good
  - More complex cryptographic applications may use higher $q = 2^{64}$, and $n = 2^{12} = 4096$
  - In the second part, we will see how to improve efficiency using more complex mathematics

Ajtai's one-way function (SIS)

- Parameters: $m, n, q \in \mathbb{Z}$
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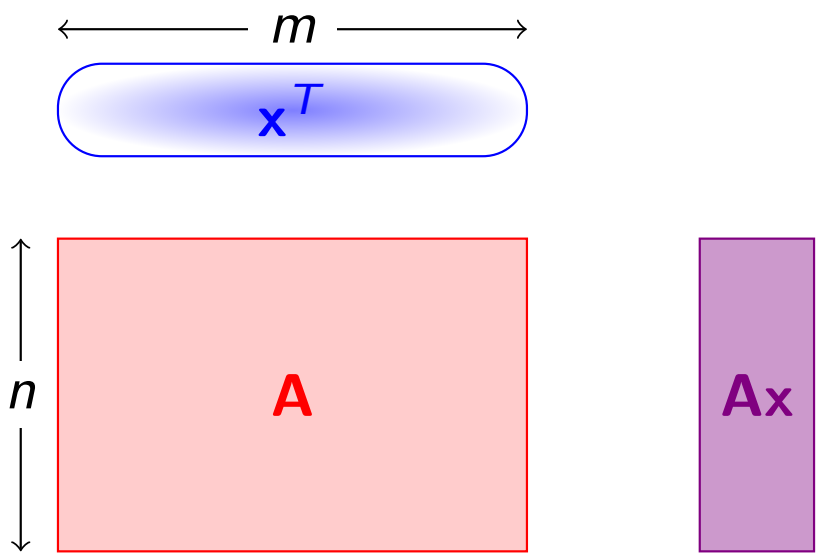

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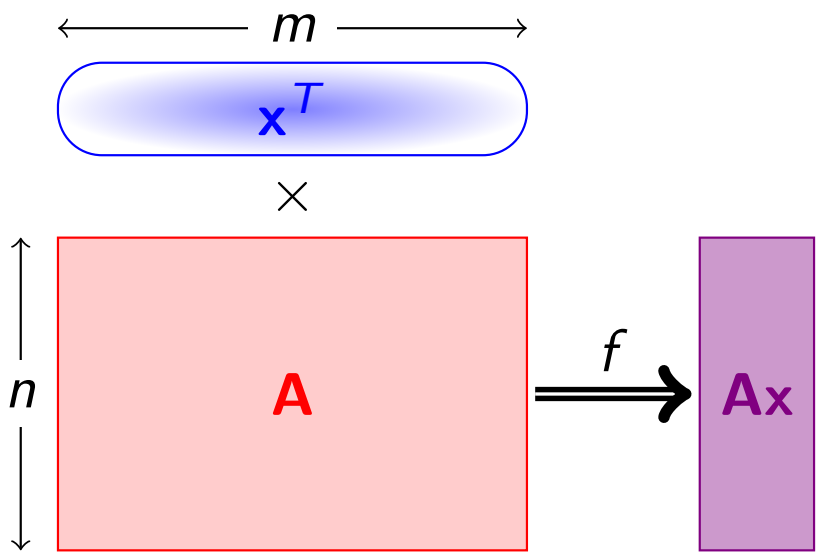

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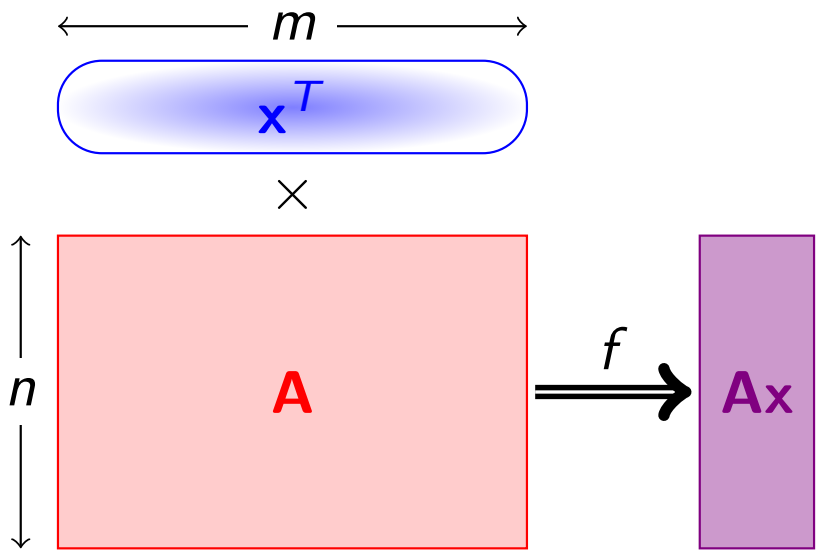


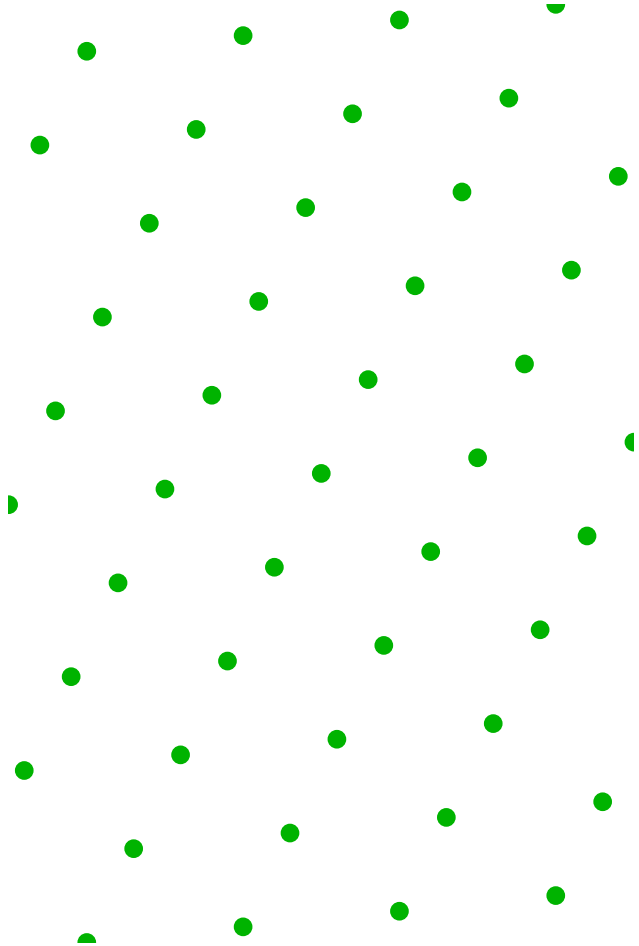
Theorem (A'96)

For $m > n \lg q$, if lattice problems (SIVP) are hard to approximate in the worst-case, then $f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q$ is a one-way function.

Applications: OWF [A'96], Hashing [GGH'97], Commit [KTX'08], ID schemes [L'08], Signatures [LM'08,GPV'08,...,DDLL'13] ...

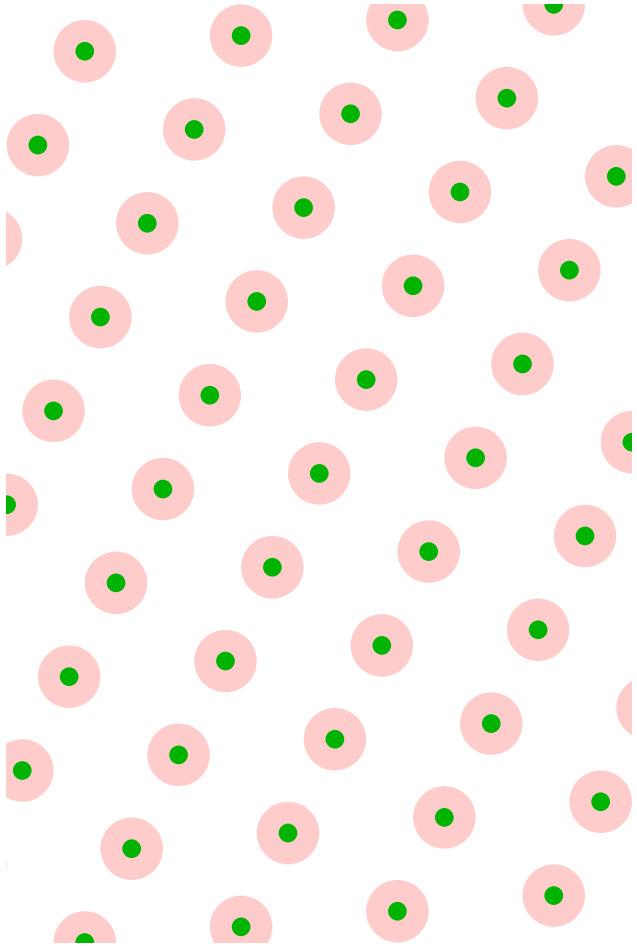
Blurring a lattice

Consider a lattice Λ, and



Blurring a lattice

Consider a lattice Λ, and add noise to each lattice point until the entire space is covered.



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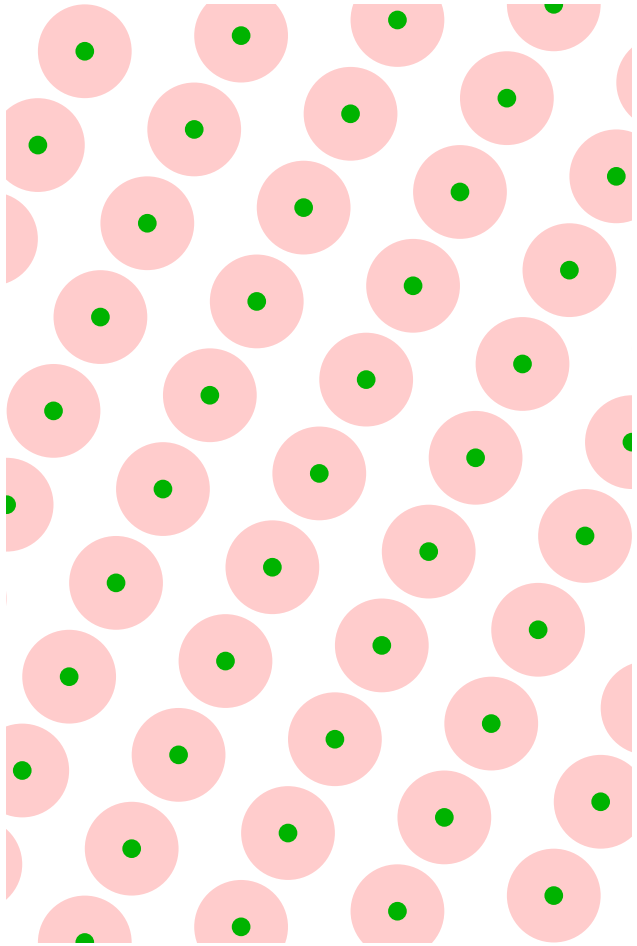

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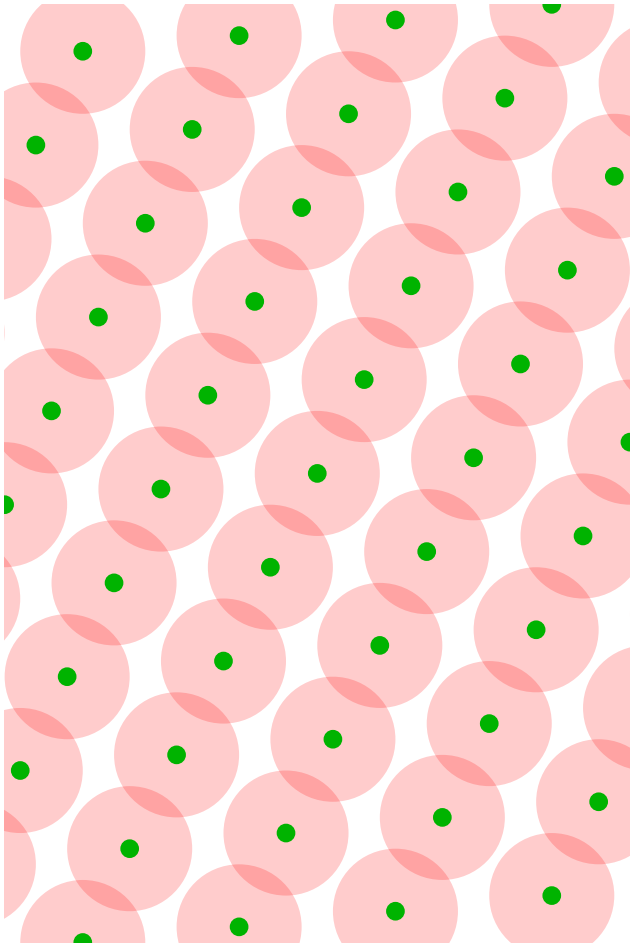

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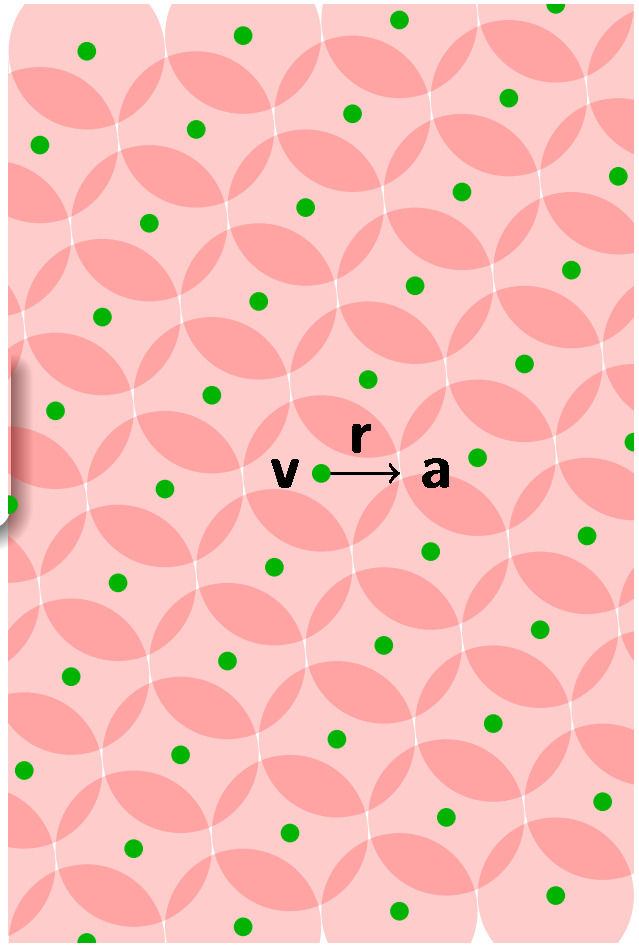

How much noise is needed?

$$\|\mathbf{r}\| \leq \sqrt{n} \cdot \lambda_n/2$$

- Each point in $\mathbf{a} \in \mathbb{R}^n$ can be written $\mathbf{a} = \mathbf{v} + \mathbf{r}$ where $\mathbf{v} \in \mathcal{L}$ and $\|\mathbf{r}\| \approx \sqrt{n}\lambda_n$.

Blurring a lattice

Consider a lattice Λ, and add noise to each lattice point until the entire space is covered. Increase the noise until the space is uniformly covered.

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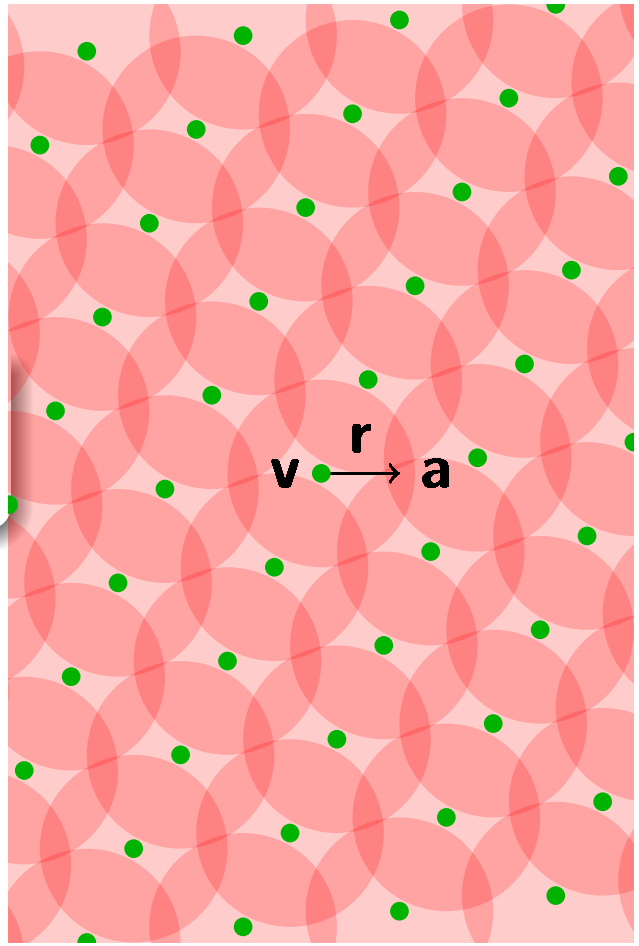

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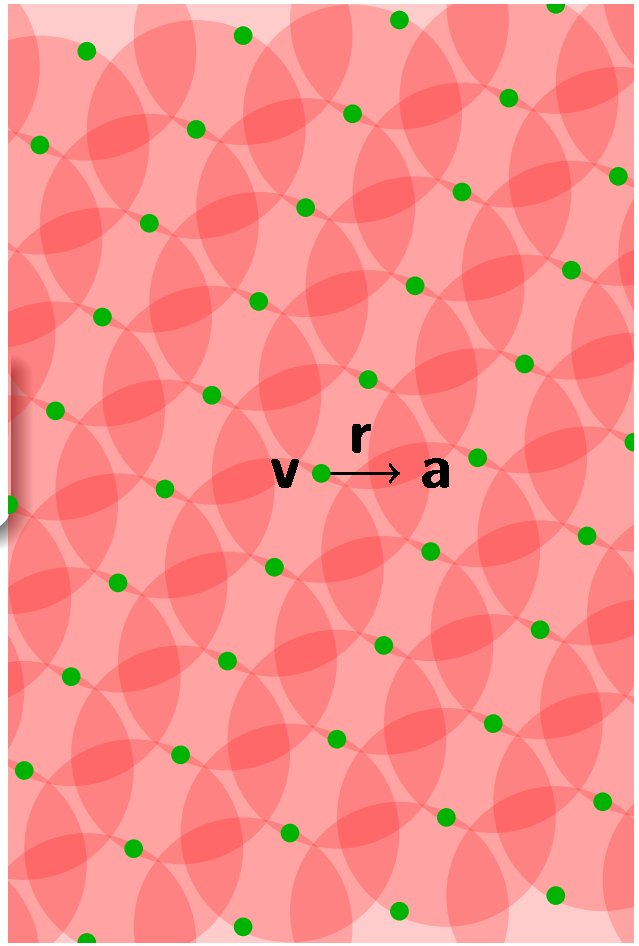

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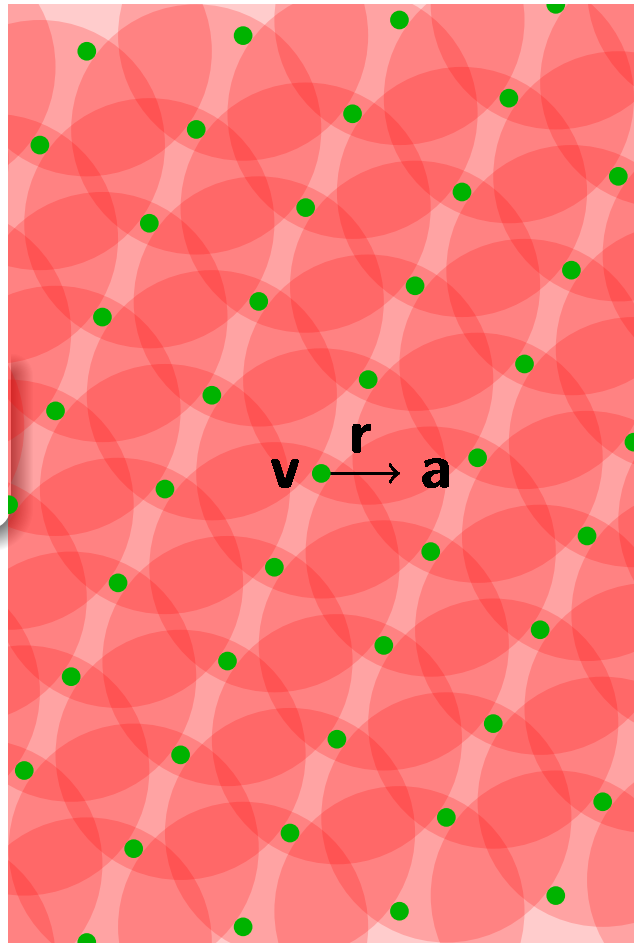

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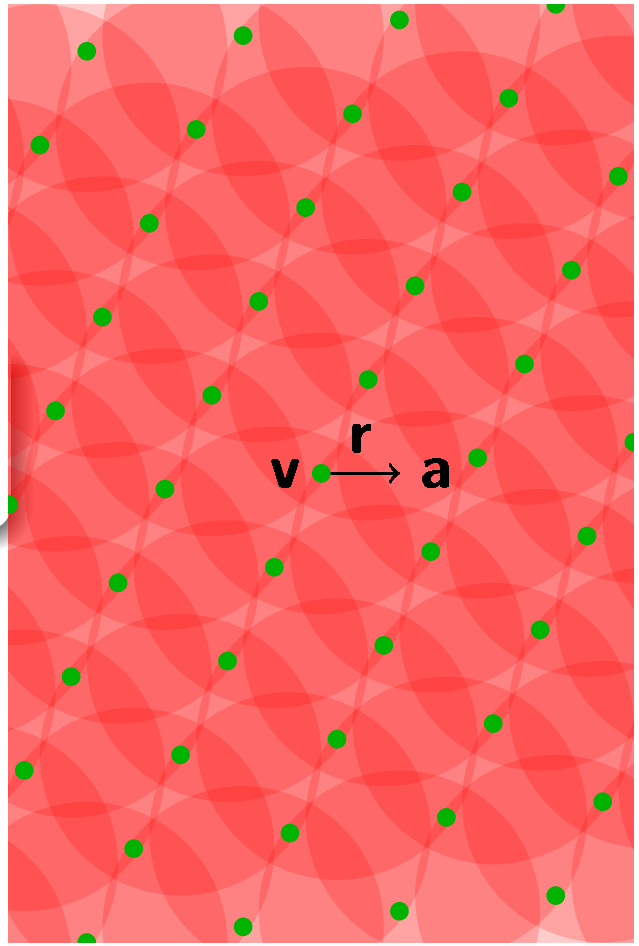

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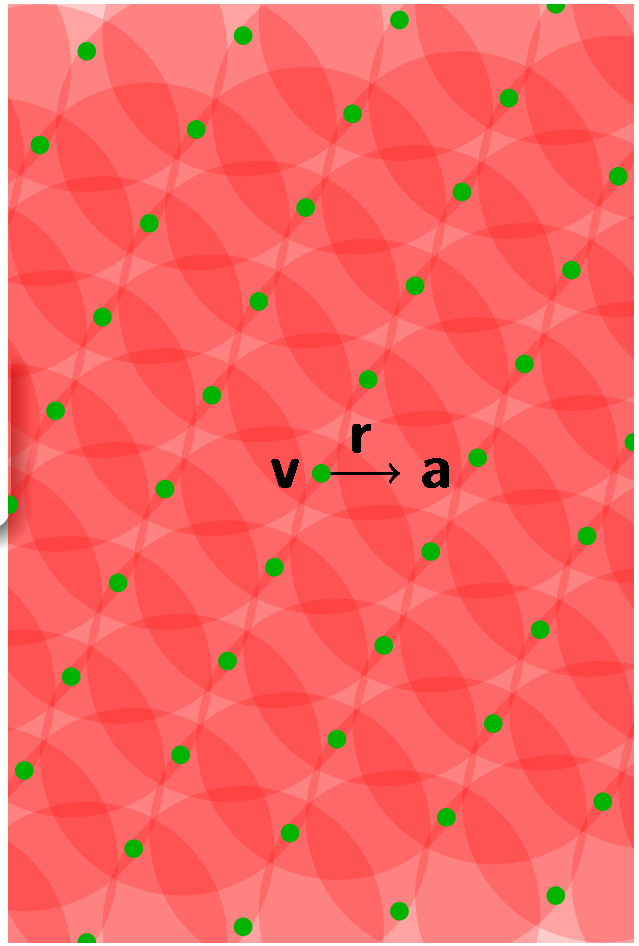

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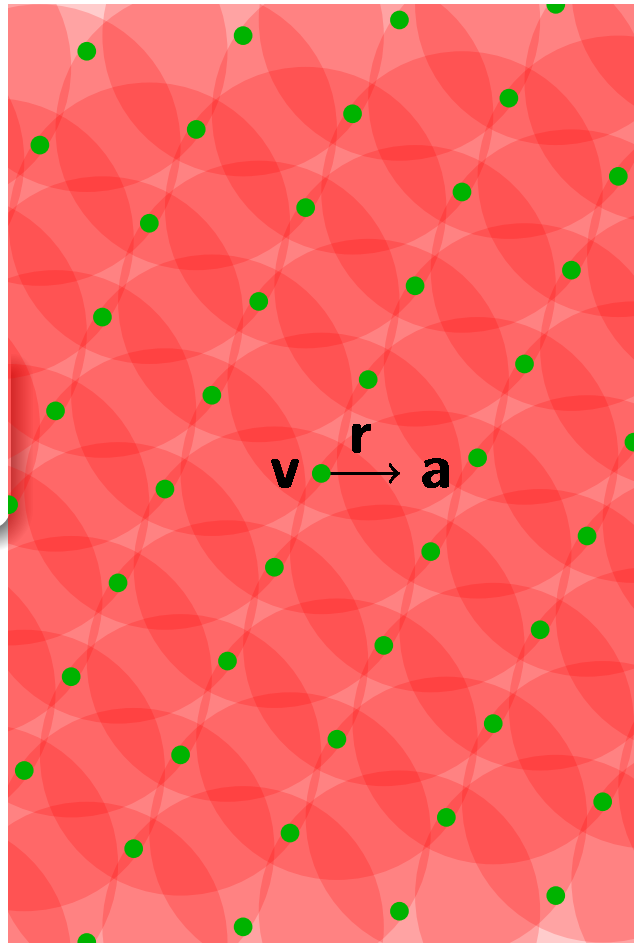
- Think of $\mathbb{R}^n \approx \frac{1}{q}\Lambda$ [GPV'07]



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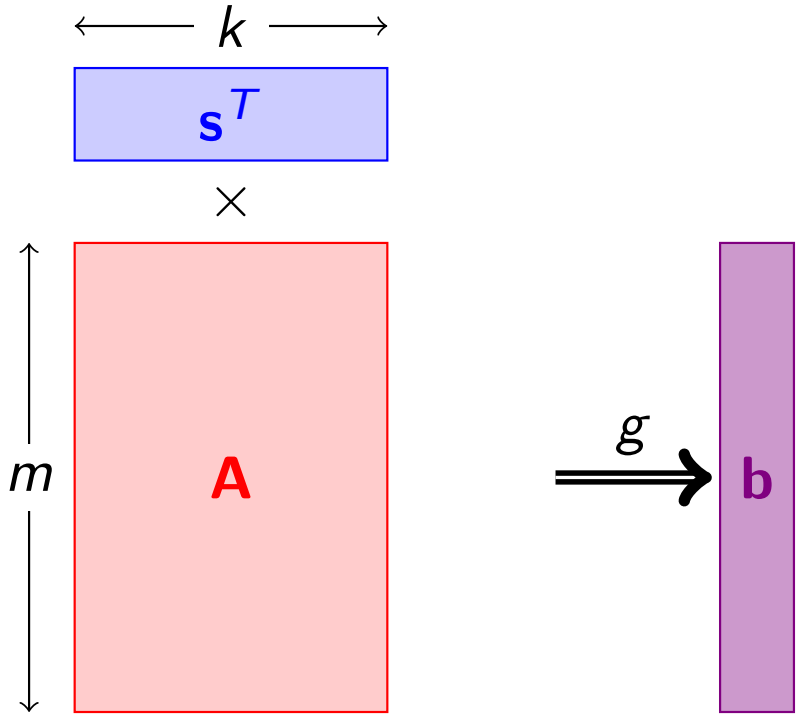
- Rearranging the terms yields a lattice vector

$$\sum \mathbf{v}_i z_i = -\sum \mathbf{r}_i z_i$$

of length at most $\| \sum \mathbf{r}_i z_i \| \approx \sqrt{m} \cdot \max \|\mathbf{r}_i\| \approx n \cdot \lambda_n$

Regev's Learning With Errors (LWE)

- $\mathbf{A} \in \mathbb{Z}_q^{m \times k}$, $\mathbf{s} \in \mathbb{Z}_q^k$, $\mathbf{e} \in \mathcal{E}^m$.
- $g_{\mathbf{A}}(\mathbf{s} \quad) = \mathbf{As} \qquad \bmod q$



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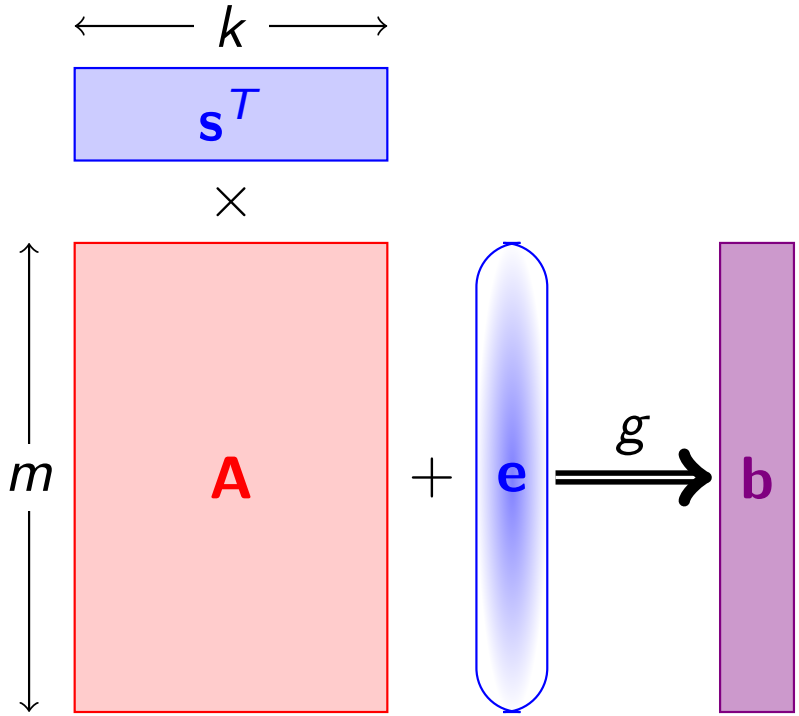

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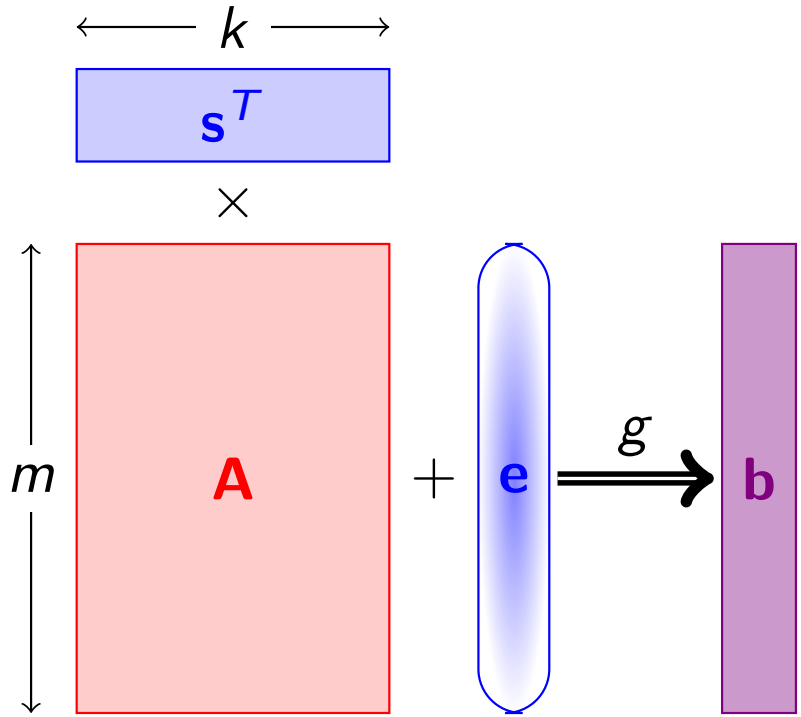

Theorem (R'05)

The function $g_{\mathbf{A}}(\mathbf{s}, \mathbf{e})$ is hard to invert on the average, assuming SIVP is hard to approximate in the worst-case.

Applications: CPA PKE [R'05], CCA PKE [PW'08], (H)IBE [GPV'08,CHKP'10,ABB'10], FHE [...,B'12,AP'13,GSW'13], ...

Hermite Normal Form (HNF)

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  - Similarly, inverting $g_{\mathbf{A}}$ is equivalent to inverting $g_{\mathbf{AU}}$

Equivalence of SIS and LWE

- Equivalent HNF variant of SIS/LWE [M'01]:
  - $f'_{\mathbf{H}}(\mathbf{x}) = f_{\mathbf{A}}(\mathbf{x})$ for $\mathbf{A} = [\mathbf{I} \mid -\mathbf{H}]$, where $\mathbf{H} \in \mathbb{Z}_q^{n \times (m-n)}$

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- Similarly, $f'_{\mathbf{H}}(\mathbf{x}) \mapsto g'_{\mathbf{H}}(\mathbb{Z}_q^k, \mathbf{x}) = g'_{\mathbf{H}}(\mathbf{s}, \mathbf{x}) + g'_{\mathbf{H}}(\mathbb{Z}_q^k, \mathbf{0})$, for $\mathbf{s} = -\mathbf{x}_2$

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- One can use $f_{\mathbf{A}}$ both for SIS and LWE

SIS/LWE as CVP

Candidate OWF

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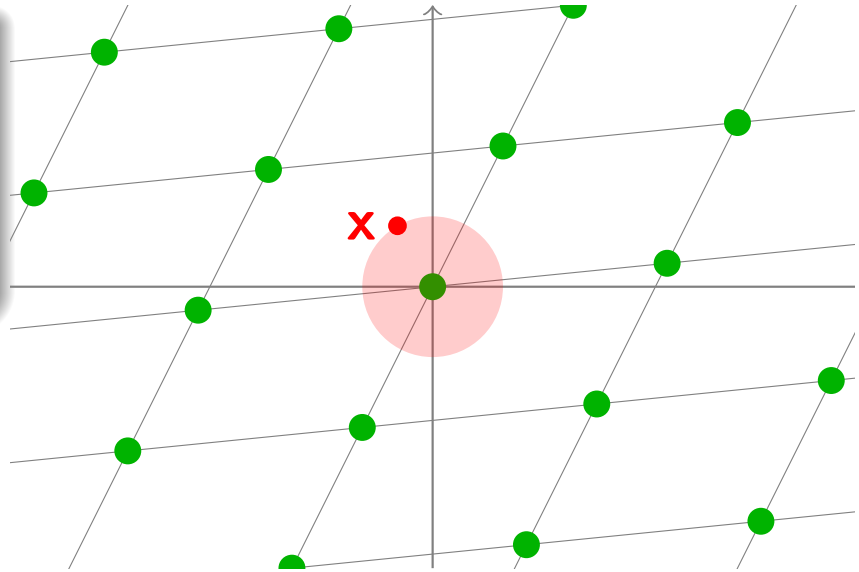

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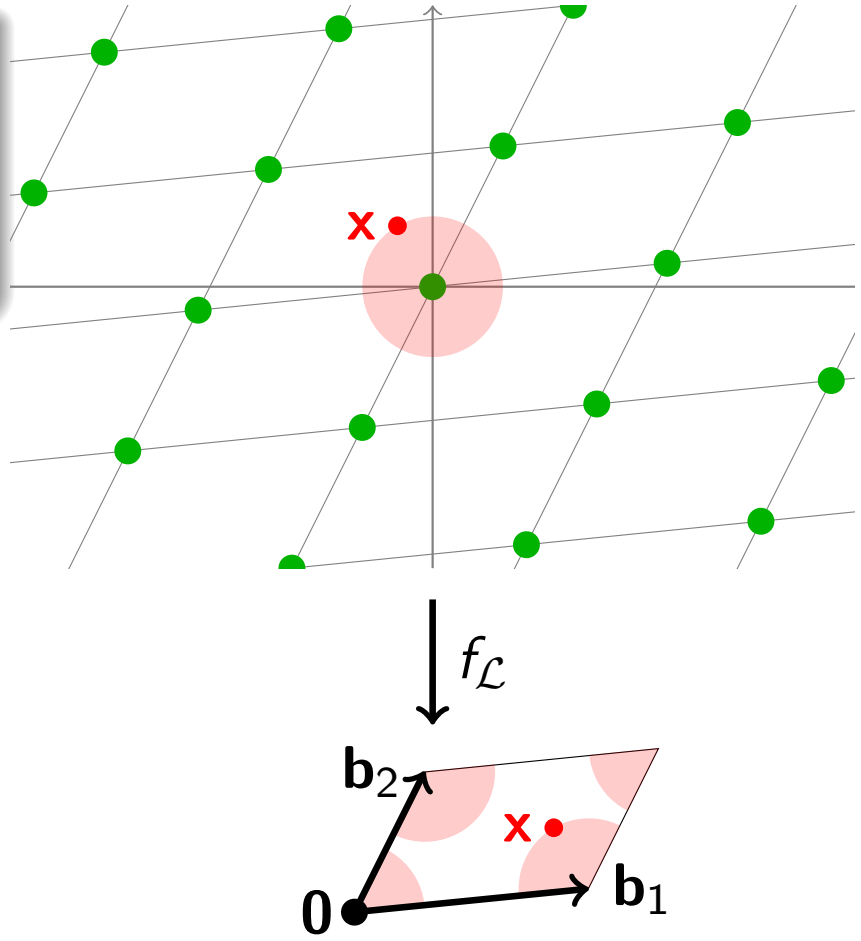

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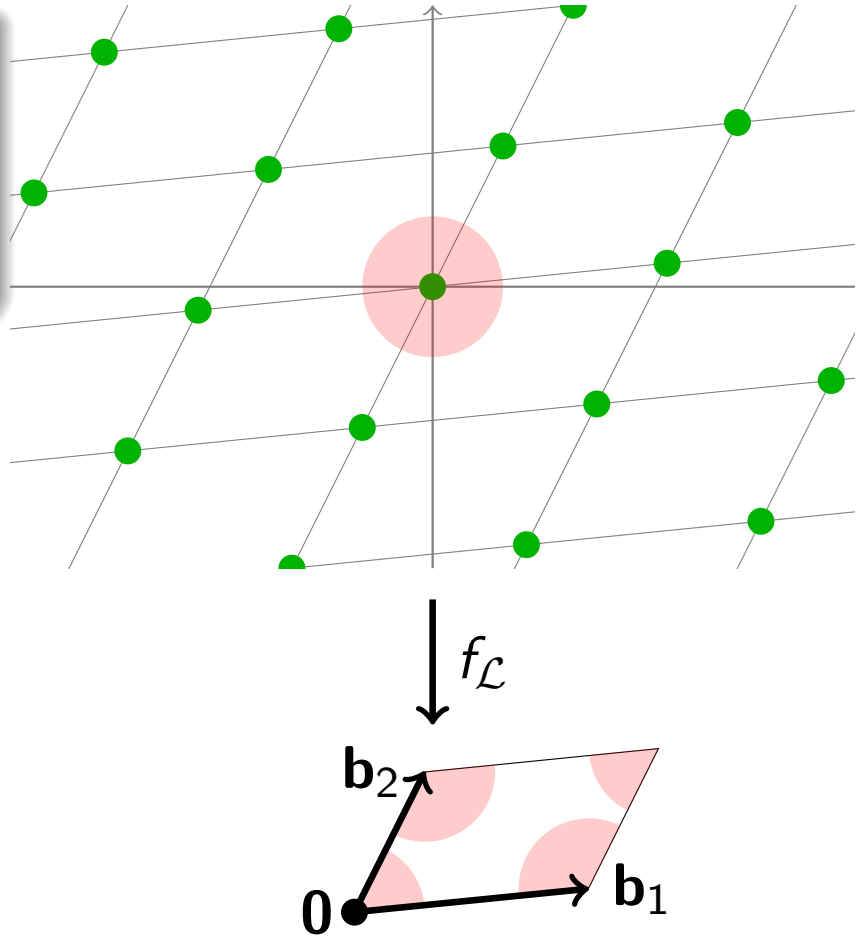

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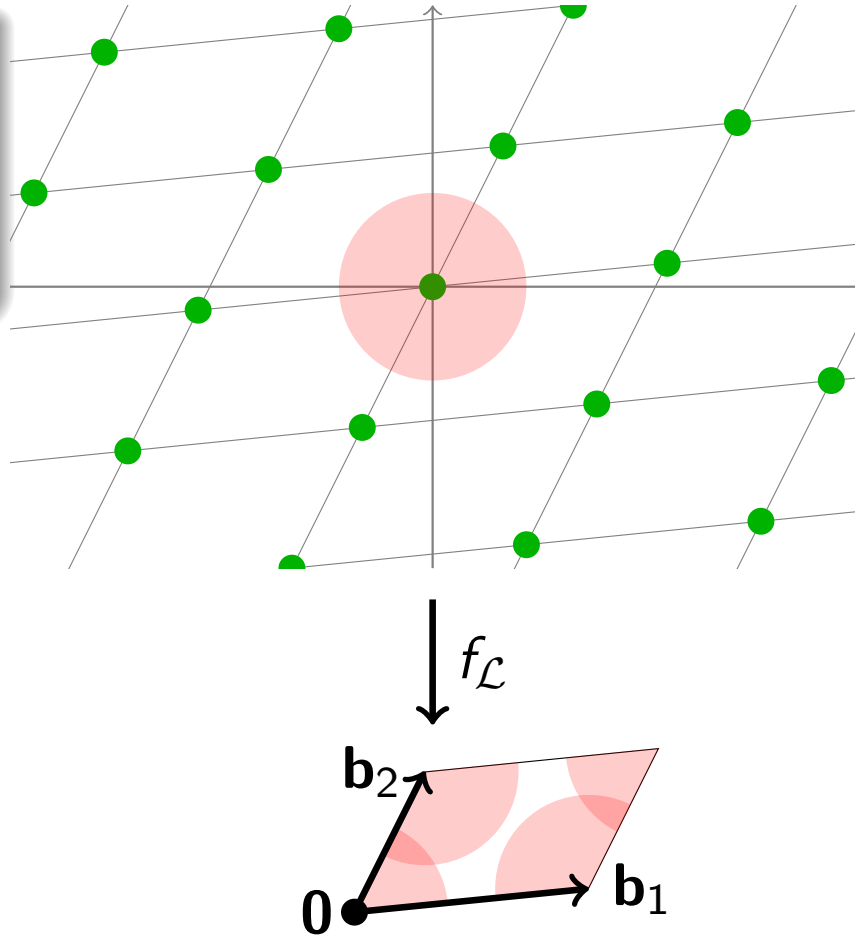

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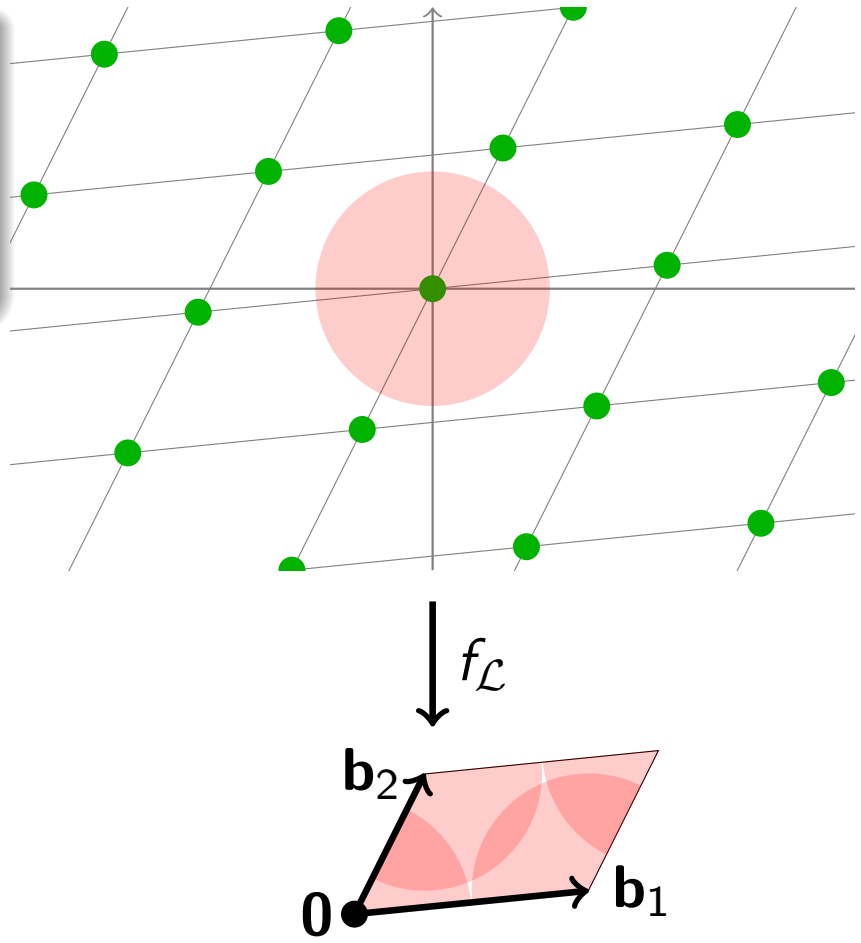

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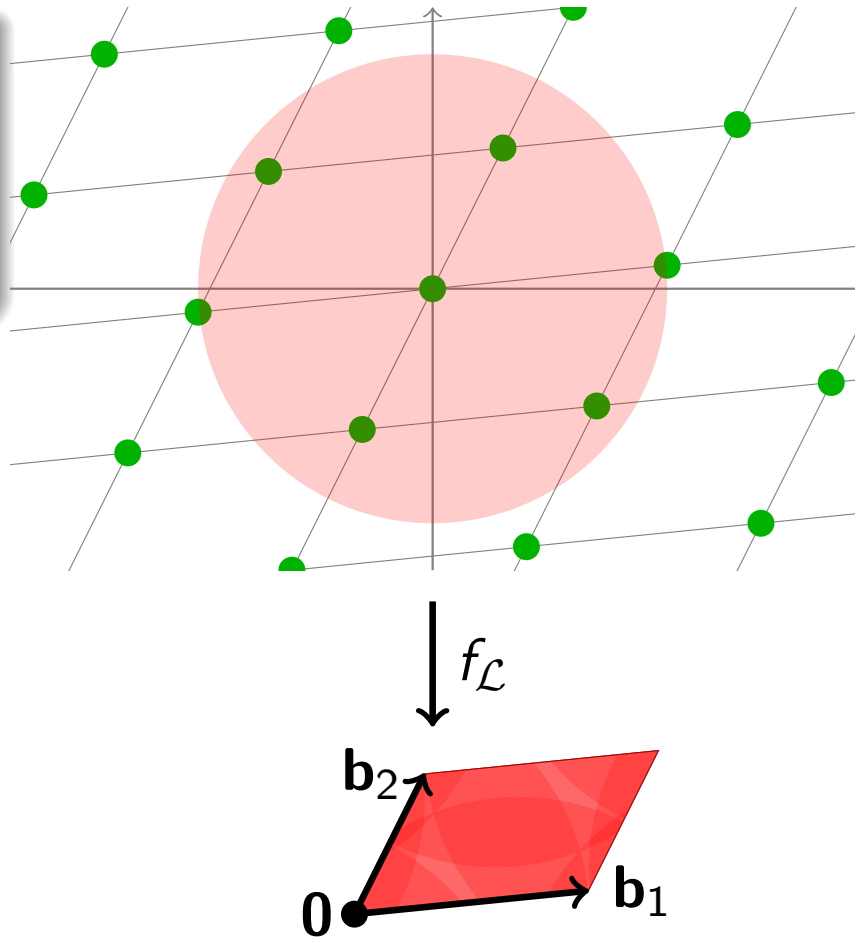
- $\beta \gg \mu$: $f_{\mathcal{L}}(\mathbf{x})$ is almost uniform



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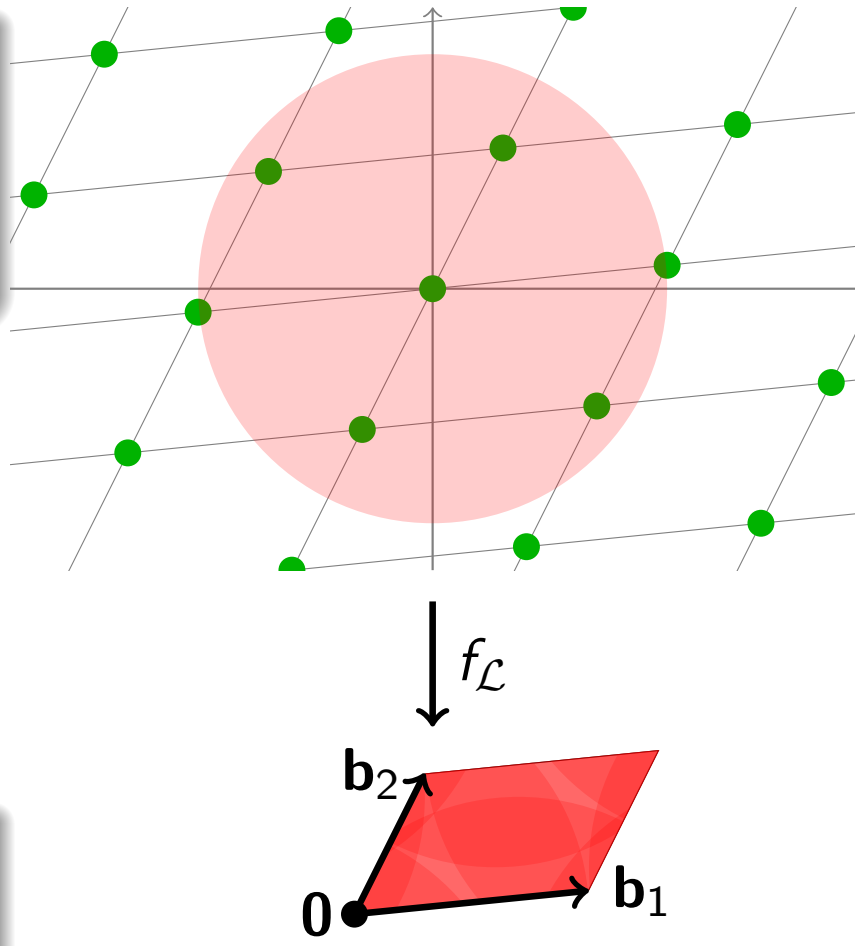
Candidate OWF

Key: a hard lattice $\mathcal{L}$
Input: $\mathbf{x}$, $\|\mathbf{x}\| \leq \beta$
Output: $f_{\mathcal{L}}(\mathbf{x}) = \mathbf{x} \bmod \mathcal{L}$

- $\beta < \lambda_1/2$: $f_{\mathcal{L}}$ is injective
- $\beta > \lambda_1/2$: $f_{\mathcal{L}}$ is not injective
- $\beta \geq \mu$: $f_{\mathcal{L}}$ is surjective
- $\beta \gg \mu$: $f_{\mathcal{L}}(\mathbf{x})$ is almost uniform

Question

Are these functions cryptographically hard to invert?



SIS Property: Regularity

$f : X \to Y$ is regular if all $y \in Y$ have same $|f^{-1}(y)|$.

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$$\mathbf{A} \in \mathbb{Z}_q^{n \times m}, \quad \mathbf{x} \in \{0,1\}^m, \quad f_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\mathbf{x} \bmod q \in \mathbb{Z}_q^n$$

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Pairwise independence:

- Fix $\mathbf{x_1} \neq \mathbf{x_2} \in \{0,1\}^m$,
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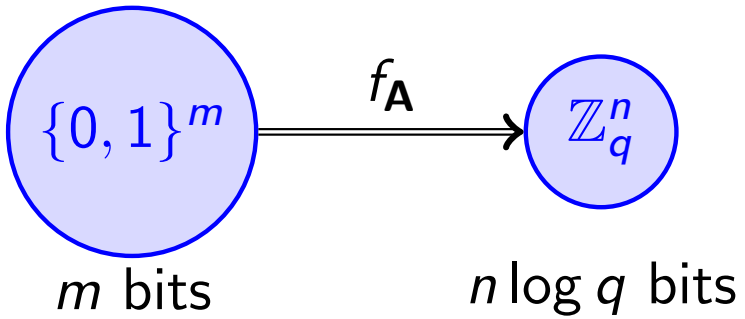
- $f_{\mathbf{A}}(\mathbf{x}_1)$ and $f_{\mathbf{A}}(\mathbf{x}_2)$ are independent.



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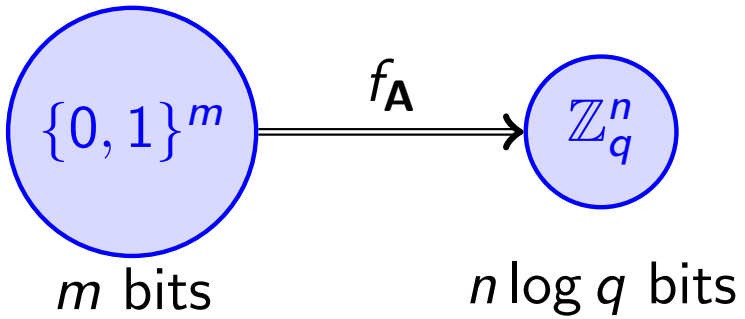

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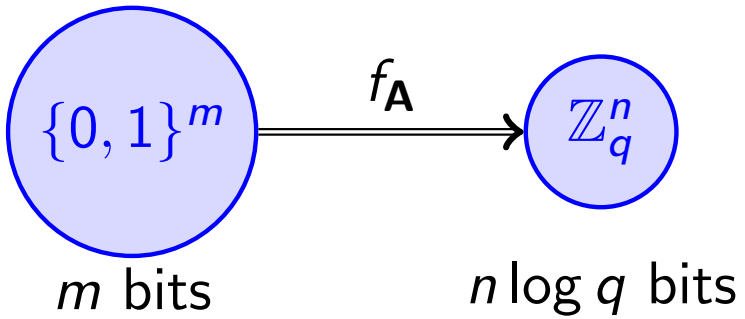

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$f_{\mathbf{A}} : (U(\{0,1\}^n)) \approx U(\mathbb{Z}_q^n)$ maps uniform to uniform.

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- Binding Property: Finding $(m, r) \neq (m', r')$ such that $C(\mathbf{m}, \mathbf{r}) = C(\mathbf{m}', \mathbf{r}')$ breaks the collision resistance of $f_{[\mathbf{A_1}, \mathbf{A_2}]}$

SIS Property: (Approximate) Linear Homomorphism

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$$f_{\mathbf{A}}(\mathbf{x}_1) + f_{\mathbf{A}}(\mathbf{x}_2) = f_{\mathbf{A}}(\mathbf{x}_1 + \mathbf{x}_2)$$

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  - Domain of $f_{\mathbf{A}}$ is not closed under $+$
- $f_{\mathbf{A}}$ is also key-homomorphic:

$$f_{\mathbf{A}_1}(\mathbf{x}) + f_{\mathbf{A}_2}(\mathbf{x}) = f_{\mathbf{A}_1 + \mathbf{A}_2}(\mathbf{x})$$

SIS Application: One-Time Signatures

- Extend $f_{\mathbf{A}}$ to matrices $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_l]$:

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- Key Generation:
  - Public Parameter: SIS function key $\mathbf{A}$
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- $Sign(sk, \mathbf{m}) = \mathbf{Xm} + \mathbf{x}$, linear combination of secret key
- $Verify(pk, \mathbf{m}, \sigma)$ uses homomorphic properties to check that

$$f_{\mathbf{A}}(\sigma) = f_{\mathbf{A}}(\mathbf{Xm} + \mathbf{x}) = f_{\mathbf{A}}(\mathbf{X})\mathbf{m} + f_{\mathbf{A}}(\mathbf{x}) = \mathbf{Ym} + \mathbf{y}$$

Pseudorandomness of LWE

LWE(k, q, m, χ) distribution: $[\mathbf{A}, \mathbf{b}] \in \mathbb{Z}_q^{m \times (k+1)}$ where

- $\mathbf{A} \leftarrow \mathbb{Z}_q^{m \times k}$
- $\mathbf{s} \leftarrow \mathbb{Z}_q^k$, $\mathbf{e} \leftarrow \chi^m$
- $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}$

Definition (Search LWE)

Given $[\mathbf{A}, \mathbf{b}]$ recover $\mathbf{s}$ and $\mathbf{e} = \mathbf{b} - \mathbf{A}\mathbf{s}$

Definition (Decision LWE)

Distinguish $[\mathbf{A}, \mathbf{b}]$ from the uniform distribution over $\mathbb{Z}_q^{m \times (k+1)}$

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Theorem (Search to Decision Reduction (informal))

For a wide range of parameters, if Search LWE is hard, then Decision LWE is hard.

Encrypting with LWE

Idea: if $[\mathbf{A}, \mathbf{b}]$ is pseudorandom, then we can use it to mask a message
Private Key Encryption

- $\mathtt{Gen}(k)$: output random $\mathbf{s} \leftarrow \mathbb{Z}_q^k$
- $\mathtt{Enc}_{\mathbf{s}}(\mathbf{m})$:
  - Choose $\mathbf{A} \leftarrow \mathbb{Z}_q^{m \times k}$ and $\mathbf{e} \leftarrow \chi^m$ at random
  - Let $\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} \pmod{q}$
  - Output $[\mathbf{A}, \mathbf{b} + \mathbf{m}]$

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  - Let $\mathbf{b} = \mathbf{As} + \mathbf{e} \pmod q$
  - Output $[\mathbf{A}, \mathbf{b} + \mathbf{m}]$
- $\mathtt{Dec}'_{\mathbf{s}}([\mathbf{A}, \mathbf{c}]) = \mathbf{c} - \mathbf{As} = \mathbf{m} + \mathbf{e}$
- Problems:
  - Decryption is "approximately" correct [CKKS 2016]
  - A passive adversary can completely break the scheme [Li,M. 2021]
- Solution: encode $\mathbf{m}$ with an error correcting code

Regev (private key) encryption

- Plaintext modulus: $p \ll q$
- Message: $\mathbf{m} \in \mathbb{Z}_p^m$
- Scaling factor: $\Delta = \left\lceil \frac{q}{p} \right\rfloor$
- $\mathtt{Enc}_{\mathbf{s}}(\mathbf{m}) = [\mathbf{A}, \mathbf{b} + \Delta\mathbf{m}]$ where $\mathbf{b} = \mathbf{As} + \mathbf{e} \pmod{q}$
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Theorem (Correctness)

If $\|\chi\|_\infty < \Delta/2$ *then*

$$\mathtt{Dec}_{\mathbf{s}}(\mathtt{Enc}_{\mathbf{s}}(\mathbf{m})) = \left\lceil \frac{(\mathbf{As} + \mathbf{e} + \Delta m) - \mathbf{As}}{\Delta} \right\rfloor = \mathbf{m} + \left\lceil \frac{\mathbf{e}}{\Delta} \right\rfloor = \mathbf{m}.$$

Additive Homomorphism

The sum of two encryption

$$\begin{aligned} \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_1) &= [\mathbf{A}_1, \mathbf{A}_1\mathbf{s} + \mathbf{e}_1 + \frac{q}{p}\mathbf{m}_1] \\ \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_2) &= [\mathbf{A}_2, \mathbf{A}_2\mathbf{s} + \mathbf{e}_2 + \frac{q}{p}\mathbf{m}_2] \end{aligned}$$

is an encryption of the sum

$$\mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_1) + \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_2) = [\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e} + \frac{q}{p}\mathbf{m}] = \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_1 + \mathbf{m}_2)$$

where

- $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 \pmod q$
- $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 \pmod q$
- $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 \pmod p$

Error growth

When adding two encryptions

$$\mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_1; \mathbf{e}_1) + \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_2; \mathbf{e}_2) = \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}_1 + \mathbf{m}_2; \mathbf{e}_1 + \mathbf{e}_2)$$

with small errors $\|\mathbf{e}_1\|, \|\mathbf{e}_2\| \leq \beta$, the result is an encryption of $\mathbf{m}_1 + \mathbf{m}_2$ with slightly larger error

$$\|\mathbf{e}_1 + \mathbf{e}_2\| \leq \|\mathbf{e}_1\| + \|\mathbf{e}_2\| \leq 2\beta.$$

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Remarks:

- If $\mathbf{e}_1, \mathbf{e}_2$ are random and independent, their sum grows more like $\sqrt{2}\beta$
- If $|\chi| \ll (q/p)$, we can add several ciphertexts, and the results will still decrypt correctly
- If we keep adding ciphertexts, the error may become too big, and decryption will fail

Linear/Affine functions?

Start from a fresh ciphertext with small $\|\mathbf{e}\|_\infty \leq \beta \ll \Delta$

$$\mathtt{Enc}_{\mathbf{s}}(\mathbf{m}) \quad = \quad [\mathbf{A}, \mathbf{As} + \mathbf{e} + \Delta\mathbf{m}]$$

What about multiplying ciphertexts by a constant or computing linear or affine function?

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t \cdot \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}) &= [(t\mathbf{A}), (t\mathbf{A})\mathbf{s} + t\mathbf{e} + \Delta(t\mathbf{m})] = \mathtt{Enc}_{\mathbf{s}}(t\mathbf{m}) \\
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&= \mathtt{Enc}_{\mathbf{s}}(\mathbf{T}\mathbf{m} + \mathbf{c})
\end{aligned}$$

Remarks:

- It works, but error grows to $\|\mathbf{T}\mathbf{e}\|_\infty \leq m \cdot \|\mathbf{T}\|_\infty \cdot \beta$
- We need $\mathbf{T}$ to be small for final result to be correct

Public Key Encryption (Regev 2005)

We will use homomorphic properties to transform private key encryption to public key encryption

- Start from LWE private key encryption scheme $(\mathtt{Gen}, \mathtt{Enc}, \mathtt{Dec})$
- Set parameters n, k, q, χ so to support the addition of w ciphertexts

Public Key Encryption scheme $(\mathtt{Gen}', \mathtt{Enc}', \mathtt{Dec})$:

- Public key: $\mathbf{P} = \mathtt{Enc}_{\mathbf{s}}(\mathbf{0})$
- $\mathtt{Enc}'_{\mathbf{P}}(\mathbf{m}) = [\mathbf{O}, \Delta\mathbf{m}] + \mathbf{R} \cdot \mathbf{P}$ where $\mathbf{R} \leftarrow \{0,1\}^{w' \times w}$

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Theorem (Correctness)

$$Enc'_{\mathbf{P}}(\mathbf{m}) = [\mathbf{O}, \Delta\mathbf{m}] + \mathbf{R}\mathtt{Enc}_{\mathbf{s}}(\mathbf{0}) = \mathtt{Enc}_{\mathbf{s}}(\mathbf{m} + \mathbf{R} \cdot \mathbf{0}) = \mathtt{Enc}_{\mathbf{s}}(\mathbf{m})$$

Theorem (Security)

If w is large enough, $\mathtt{Enc}'_{\mathbf{P}}(\mathbf{m}) \approx \mathtt{Enc}_{\mathbf{s}}(\mathbf{m}; \chi')$.

Optimized Public Key Encryption (Lindner,Peikert 2011)

- LWE PKE = "Symmetric Encryption + Linearity"
- We can also describe it directly in terms of matrices:
  - Public Key: $\mathbf{P} = g_{\mathbf{A}}(\mathbf{S}, \mathbf{E}) = [\mathbf{A}, \mathbf{B} = \mathbf{AS} + \mathbf{E}]$
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- We can use a smaller dimensional $\mathbf{A}, \mathbf{P}$ (and improve efficiency) using the SIS/HNF variant of LWE:
  - $\mathbf{A}, \mathbf{S}, \mathbf{E}, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in \mathbb{Z}_q^{k \times k}$
  - Public Key: $\mathbf{P} = [\mathbf{A}, f_{[\mathbf{A},\mathbf{I}]}(\mathbf{S}, \mathbf{E})] = [\mathbf{A}, \mathbf{B} = \mathbf{AS} + \mathbf{E}] \in \mathbb{Z}_q^{k \times 2k}$
  - Expanded public key: $[\mathbf{P}^t, \mathbf{I}] \in \mathbb{Z}_q^{2k \times 3k}$

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  - Public Key: $\mathbf{P} = [\mathbf{A}, f_{[\mathbf{A},\mathbf{I}]}(\mathbf{S}, \mathbf{E})] = [\mathbf{A}, \mathbf{B} = \mathbf{AS} + \mathbf{E}] \in \mathbb{Z}_q^{k \times 2k}$
  - Expanded public key: $[\mathbf{P}^t, \mathbf{I}] \in \mathbb{Z}_q^{2k \times 3k}$
  - Encryption (of 0):

$$\mathtt{Enc}_{\mathbf{P}}(0)^t = f_{[\mathbf{P}^t,\mathbf{I}]}(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) = \mathbf{P}^t\mathbf{R}_1 + \begin{bmatrix} \mathbf{R}_2 \\ \mathbf{R}_3 \end{bmatrix} \in \mathbb{Z}_q^{2k \times k}$$

- This is the blueprint followed by the ML-KEM (Kyber) cryptosystem standardized by NIST, but with $\mathbb{Z}_q^{k \times k}$ replaced by more compact "algebraically structured" matrices

Gadgets

Questions:

- Can we encrypt messages $\mathbf{m} \in \mathbb{Z}_q$ with plaintext modulus $p = q$?
- Can we multiply ciphertext by any $t \in \mathbb{Z}_q$ with error growth $\ll q$?

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2. Given $\mathbf{c} = \mathbf{g} \cdot m + \mathbf{e}$ for some $\|\mathbf{e}\|_\infty < q/4$, we can recover $m \in \mathbb{Z}_q$ (written $m = \lceil \mathbf{c} \rfloor_{\mathbf{g}}$)

The "Gadget-LWE" encryption scheme

Similar to LWE, but using the gadget vector $\mathbf{g}$ instead of scaling factor Δ to encode the message.

- Plaintext modulus $p = q$
- $\mathtt{Enc}_{\mathbf{s}}^{\mathbf{g}}(m) = [\mathbf{A}, \mathbf{As} + \mathbf{e} + \mathbf{g} \cdot m]$
- $\mathtt{Dec}_{\mathbf{s}}^{\mathbf{g}}([A, \mathbf{c}]) = \lceil \mathbf{c} - \mathbf{As} \rfloor_{\mathbf{g}}$

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Bonus: we can multiply ciphertexts by arbitrary constants (or linear transformations)

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Similarly: $(\mathbf{g} \cdot c) \odot \mathtt{Enc}_{\mathbf{s}}^{\mathbf{g}}(m) = \mathtt{Enc}_{\mathbf{s}}(\mathbf{g}cm) = \mathtt{Enc}_{\mathbf{s}}^{\mathbf{g}}(cm)$
Remark: error grows by $\ell = \log_2 q$ instead of q because $\mathbf{g}^{-1}(c) \in \{0,1\}^{\ell}$

Other gadgets

- The "powers-of-two" gadget is just the most common example.
- Easily generalize to arbitrary base:
  $\mathbf{g}^t = [1, B, B^2, \ldots, B^{\log_B q - 1}] \in \mathbb{Z}^\ell$, gives a trade-off between storage $\ell = \log_B q$ and error growth $B \cdot \ell$

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- Another useful option is the "Chinese Remainder Theorem" (CRT) gadget $\mathbf{g}^t = [q/p_1, \ldots, q/p_\ell]$ where $q = \prod_i p_i$
- You can even combine a scaling factor $\Delta = q/p$ and a gadget $\mathbf{g}$ (mod p), and encode $m \in \mathbb{Z}_p$ as $\Delta \cdot (\mathbf{g}m) \in \mathbb{Z}_q^\ell$.
- Many optimizations and variants in lattice-based cryptography can be seen simply as a different choice of gadget $\mathbf{g}$

Application: Key-Switching

The key-switching problem: $\mathtt{Enc}^{\mathbf{g}}_{\mathbf{s}}(m) \mapsto \mathtt{Enc}^{\mathbf{g}}_{\mathbf{z}}(m)$

- You have a ciphertext $\mathbf{C} = \mathtt{Enc}^{\mathbf{g}}_{\mathbf{s}}(m) = [\mathbf{A}, \mathbf{b}]$ under a key $\mathbf{s}$
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So far: summary

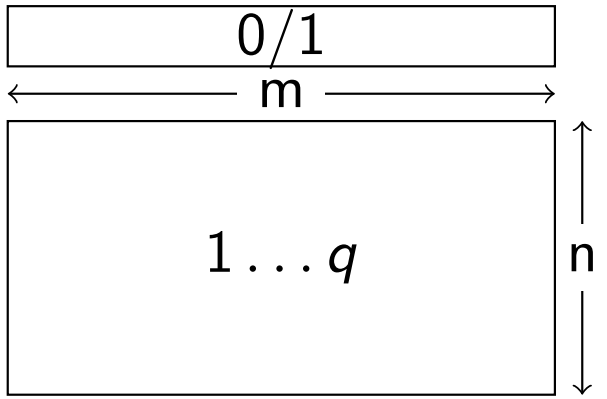
- SIS, LWE: hard lattice problems on random q-ary lattices $\Lambda_q(\mathbf{A})$
- Technical tool: Lattice gadgets
- Strong linear homomorphic properties
- Typical applications:
  - SIS: collision resistant hashing
  - LWE: encryption
- Main issue: Efficiency!

Part 2: Algebraic Lattices

- Structured matrices and Polynomial Rings
- RingSIS and RingLWE
- Algorithms: FFT and NTT
- Example: Kyber (ML-KEM)

Efficiency of Ajtai's function

- $q = n^{O(1)}$, $m = O(n \log n) > n \log_2 q$
- E.g., $n = 64$, $q = 2^8$, $m = 1024$
- $f_{\mathbf{A}}$ maps 1024 bits to 512.



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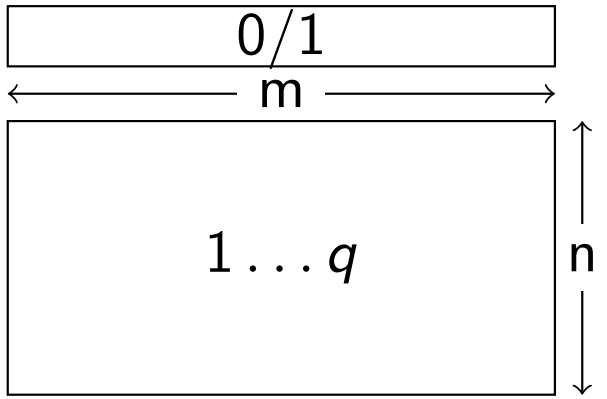

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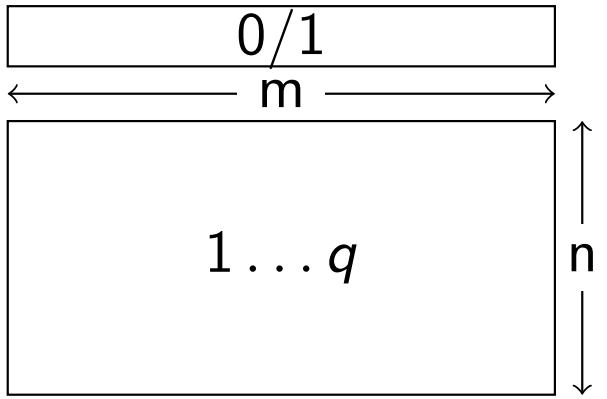
- Usable, but inefficient
  - Source of inefficiency: quadratic dependency in n



Problem

Can we do better than $O(n^2)$ complexity?

Efficient lattice based hashing

Idea: use structured matrix $\mathbf{A} = [\mathbf{A}^{(1)} \mid \ldots \mid \mathbf{A}^{(m/n)}]$, $\quad \mathbf{A}^{(i)} \in \mathbb{Z}_q^{n \times n}$.

Theorem (Micciancio 2002)

$f_{\mathbf{A}}$ is a one-way function, provided SVP and SIVP are hard to approximate within $\gamma \approx n$ in the worst case over cyclic lattices

$$\mathbf{A}^{(i)} = \begin{bmatrix} a_1^{(i)} & a_n^{(i)} & \cdots & a_2^{(i)} \\ a_2^{(i)} & a_1^{(i)} & \cdots & a_3^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(i)} & a_{n-1}^{(i)} & \cdots & a_1^{(i)} \end{bmatrix}$$

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Can you find a collision? (mod 10)

1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3

Can you find a collision? (mod 10)

1	0	0	-1	-1	1	1	0	0	0	1	1	1	0	-1	0	
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1	5
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7	4
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2	8
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3	6

Can you find a collision? (mod 10)

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1	0
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7	0
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2	0
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3	0

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1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1		
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7		
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2		
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3		

6
6
6
6

9
9
9
9

7
7
7
7

3
3
3
3

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1	1	1	1	-1	-1	-1	-1	0	0	0	0	1	1	1	1		
1	4	3	8	6	4	9	0	2	6	4	5	3	2	7	1		0
8	1	4	3	0	6	4	9	5	2	6	4	1	3	2	7		0
3	8	1	4	9	0	6	4	4	5	2	6	7	1	3	2		0
4	3	8	1	4	9	0	6	6	4	5	2	2	7	1	3		0

$$+1 \times \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} \quad -1 \times \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix} \quad +0 \times \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix} \quad +1 \times \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$$

Remarks about proofs of security

- This function is essentially the compression function of hash function LASH, modeled after NTRU
- You can still "prove" security based on average case assumption: Breaking the above hash function is as hard as finding short vectors in a random lattice $\Lambda([\mathbf{A}^{(1)}|\ldots|\mathbf{A}^{(m/n)}])$
- . . . but we know the function is broken: The underlying random lattice distribution is weak!
- Conclusion: Assuming that a problem is hard on average-case is a really tricky business!

Can you find a collision now? (mod 10)

?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
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?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
1	-4	-3	-8	6	-4	-9	-0	2	-6	-4	-5	3	-2	-7	-1
8	1	-4	-3	0	6	-4	-9	5	2	-6	-4	1	3	-2	-7
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Theorem (trivial)

Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices

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?	?	?	?	?	?	?	?	?	?	?	?	?	?	?	?
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Theorem (trivial)

Finding collisions on the average is at least as hard as finding short vectors in the corresponding random lattices

Theorem (LM'07,PR'07)

Provably collision resistant, assuming the worst case hardness of approximating SVP and SIVP over anti-cyclic lattices.

Polynomial rings

- $\mathbb{Z}[X]$: set of polynomials with integer coefficients
- Monic polynomial $f(X) = X^d + f_{d-1}X^{d-1} + \cdot + f_1 \cdot X + f_0 \in \mathbb{Z}[X]$
- $a(X) \pmod{f(X)}$ has degree $< d$
- $R = \mathbb{Z}[X]/f(X) \equiv \mathbb{Z}^d$

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- $a(X) \pmod{f(X)}$ has degree $< d$
- $R = \mathbb{Z}[X]/f(X) \equiv \mathbb{Z}^d$
- Matrix representation

$$M_{a(X)} = \left[(a(X)), (X \cdot a(X)), \ldots, (X^{d-1} \cdot a(X))\right] \in \mathbb{Z}^{d \times d}$$

  where $X^i \cdot a(X)$ is reduced $\pmod{f(X)}$
- $c(X) = a(X) \cdot b(X) \bmod f(X)$ then

$$M_{a(X)} \cdot b(X) = c(X)$$

$$M_{a(X)} \cdot M_{b(X)} = M_{c(X)}$$

Examples

- If $f(X) = X^d - 1$, then $X^d \equiv 1$ and

$$M_a = \begin{bmatrix} a_0 & a_{d-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-1} & a_{d-2} & \cdots & a_0 \end{bmatrix}$$

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- If $f(X) = X^d + 1$, then $X^d \equiv -1$ and

$$M_a = \begin{bmatrix} a_0 & -a_{d-1} & \cdots & -a_1 \\ a_1 & a_0 & \cdots & -a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-1} & a_{d-2} & \cdots & a_0 \end{bmatrix}$$

- What makes $X^d + 1$ better than $X^d - 1$?

Choosing $f(X)$

- $f(X) = X^d - 1$
  - Factors into $X^d - 1 = (X - 1) \cdot (X^{d-1} + X^{d-2} + \cdots X + 1)$
  - Ability to find collisions is closely related to existence of linear factor $(X - 1)$

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  - Efficient, easier to implement
  - These are called "power-of-two" cyclotomic rings
- One may use other irreducible $f(X)$
  - For example, $f(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$
  - Less convenient, harder to implement
- From now on assume $R = \mathbb{Z}[X]/(X^d + 1) \equiv \mathbb{Z}^d$ with $d = \dim(R) = 2^k$

Algebraic Number Theory

- $K = \mathbb{Q}[X]/f(X) \equiv \mathbb{Q}^d$ is called an Algebraic Number Field
  - May be defined for any irreducible $f(X) \in \mathbb{Z}[X]$
  - Here we focus on $f(X) = X^d + 1$ for $d = 2^k$
- $R \subset K$ is the "Ring of integers" of K
- Very special case: $d = 1 = 2^0$
  - $\mathbb{Q}[X]/(X+1) = \mathbb{Q}$
  - $\mathbb{Z}[X]/(X+1) = \mathbb{Z}$
- R generalizes $\mathbb{Z} \subset \mathbb{Q}$ to higher dimension $d = \dim(R)$
  - Ring element $a(X) \in R$ can be stored as vector in $\mathbb{Z}^d$
  - It represent a (structured) matrix in $\mathbb{Z}^{d \times d}$.
  - For $d = 256$ or higher, this is a huge saving in storage
- You don't need to known Algebraic Number Theory to understand Lattice-based cryptography

RingSIS

- Just like SIS, but using R instead of $\mathbb{Z}$
- Let $R_q = R/qR = \mathbb{Z}_q[X]/f(X)$
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  - For power-of-two cyclotomic rings R, just take the norm of the coefficient vector $\|\mathbf{a}\|$
- Effective dimension $\mathbf{A} \in \mathbb{Z}_q^{nd \times md}$
  - For large d, it is enough to take $n = 1$ and $m = 2n \log_2 q = 2 \log_2 q$
  - For $d = 256$, $q = 2^{16}$, now $\mathbf{A}$ only takes $64d$ $\mathtt{B} = 16$ $\mathtt{KB}$, instead of $64d^2$ $\mathtt{B} = 4$ $\mathtt{MB}$.

RingLWE

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  - Search RingLWE: Find $\mathbf{s}$ and $\mathbf{e} = \mathbf{b} - \mathbf{As}$
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- Can set $k = 1$: distinguish $[\mathbf{a}, \mathbf{a} \cdot s + \mathbf{e}_i]$ from uniform $R_q^{m \times 2}$
  - The secret $s \in R_q$ is just a ring element
  - d serves as a security parameter
- When $k > 1$, this is sometime called ModuleLWE.

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- Approach:
  - Convert ring elements $a(X) \in R$ between their "coefficient representation $[a_0, \ldots, a_{d-1}]$ and their evaluation representation

$$(a(\omega_0), \ldots, a(\omega_{d-1}))$$

  - Note that in the evaluation representation, multiplication is componentwise

$$(a \cdot b)(\omega_i) = a(\omega_i) \cdot b(\omega_i)$$

    and can be computed with d scalar multiplications

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- Compute the polynomial product $a(X) \cdot b(X) \pmod{f(X)}$ as

$$a(X) \cdot b(X) = \mathbf{fft}^{-1}(\mathbf{fft}(a) \circ \mathbf{fft}(b))$$

  where $\circ$ is componentwise multiplication.
- Total running time is $2 \cdot O(d \log d) + O(d) = O(d \log d)$ arithmetic operations on *complex numbers*

The Number Theoretic Transform (NTT)

- We want to multiply polynomials in $R_q = \mathbb{Z}_q[X]/f(X)$
- Choose prime q such that $q \equiv 1 \pmod{2d}$
  - Since $2d$ divides $q - 1 = |\mathbb{Z}_q^*|$, there is an element $\omega \in \mathbb{Z}_q^*$ of order $2d$
  - The values $\omega_j = \omega^{2j+1} \in \mathbb{Z}_q$ are roots of $f(\omega_j) = 0 \pmod{q}$
  - We can perform the FFT computation in $\mathbb{Z}_q$, using ω
- This is called the Number Theoretic Transform

$$\textbf{ntt}: \mathbb{Z}_q^d \to \mathbb{Z}_q^d$$

- The effective use of NTT to efficiently implement (Ring) lattice cryptography first introduced and popularized by the SWIFFT hash function (Lyubashevsky,M.,Peikert,Rosen, 2008)

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  - Scalars $a(X) = a_0 \in \mathbb{Z}$ are mapped to diagonal matrices $M_a = a_0 \cdot \mathbf{I}$
  - For example, the powers-of-two gadget becomes

$$\mathbf{G} = [\mathbf{I}, 2\mathbf{I}, 4\mathbf{I}, \ldots]^t$$

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  - All gadget algorithms (inversion, decomposition) work on ring elements componentwise, on the coefficients of the input polynomial $a(X) = [a_0, \ldots, a_{d-1}]$
- Error is measured in the coefficient representation:
  - Gadget operations should always be performed on the coefficient representation
  - Can use $\mathbf{ntt}$ and $\mathbf{ntt}^{-1}$ to convert between coefficient and evaluation representation

Coefficient vs Evaluation representation

- Ring elements can be equivalently represented as
  - coefficient vectors $[a_0, \ldots, a_{d-1}] \in \mathbb{Z}_q^d$, or
  - evaluation vectors $[a(\omega_0), \ldots, a(\omega_{d-1})] \in \mathbb{Z}_q^d$
- Both representations are equally compact, but
  - the coefficient representation supports the evaluation of gadget operations
  - the evaluation representation allows to perform ring arithmetics (additions and multiplications) in linear time $O(d)$
- One can convert between the two representations in time $O(d \log d)$ using $\mathbf{ntt}, \mathbf{ntt}^{-1}$, and this is often the bottleneck in practical implementations of lattice cryptography

Kyber PKE – Parameters

- Core component of ML-KEM
- Parameters:
  - Ring dimension $d = 2^8 = 256$
  - 12-bit prime modulus $q = 3329 = 13 \cdot 256 + 1$
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- Technically, a full NTT would have required $2d = 512$ to divide $q - 1$, but 256 is good enough for efficient implementation
- Ring $R_q = \mathbb{Z}_q[X]/(X^{256} + 1)$
- Storage (per ring element): $d \cdot \log_2 q = 256 \cdot 12 = 3072$ bits
- Security level: $k \cdot d = 512, 768, 1024$

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  - $\hat{\mathbf{A}} = \mathbf{ntt}(\mathbf{A}) \in R_q^{k \times k}$, generated from a short seed $\rho \in \{0,1\}^{256}$
  - $\mathbf{s}, \mathbf{e} \in R_q^k$, chosen from distribution χ_1 on small elements
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- Encryption $\mathtt{Enc}_{[\hat{\mathbf{A}}, \hat{\mathbf{b}}]}(m) = (\mathbf{u}, v)$:
  - $\mathbf{r} \leftarrow \chi_1^k$, $\mathbf{e} \leftarrow \chi_2^{k+1}$
  - $$\begin{bmatrix} \mathbf{u} \\ v \end{bmatrix} = \mathbf{ntt}^{-1}\left(\begin{bmatrix} \hat{\mathbf{A}}^t \\ \hat{\mathbf{b}}^t \end{bmatrix} \circ \mathbf{ntt}(\mathbf{r}) \right) + \mathbf{e} + \begin{bmatrix} \mathbf{0} \\ \Delta \mathbf{m} \end{bmatrix}$$

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- Decryption: $\mathtt{Dec}_{\hat{\mathbf{s}}}(\mathbf{u}, v) = \left\lceil (v - \mathbf{ntt}^{-1}(\hat{\mathbf{s}} \circ \mathbf{ntt}(\mathbf{u})))/\Delta \right\rfloor$

Conclusion

- Lattice cryptography: mature area building on +20 years of research
- Algebraic lattices: key to efficient implementation
- Tools and techniques: linear algebra, gadget lattices
- Natural homomorphic properties: rich set of applications
- Ready to use Public Key Encryption (and Digital Signatures)
- Any Questions?